PHYS 7810
Hydrodynamics
Spring 2026

Lecture 2
Fokker-Planck equation

January 13

Stochastic equation: $\frac{dx}{dt} = a(x) + b(x)\xi(t)$

Gaussian white noise:

$\langle \xi(t) \rangle = 0$

$\langle \xi(t)\xi(s) \rangle = \delta(t-s)$

Alternative: $dx = a(x)dt + b(x)d\xi$, $\langle d\xi \rangle = 0$, $\langle d\xi^2 \rangle = dt$.

Not well-posed if b depends on x.

In today's lecture we will find a nice way to address that issue. Let's start by trying to calculate something more "tangible"

probability density $P(x,t)$: $\mathbb{P}(\alpha \leq x(t) \leq \beta) = \int_\alpha^\beta dx \; P(x,t)$

probability that...

unlike $x(t)$, not a random variable!

Theorem: In Ito prescription,

$$\boxed{\partial_t P = -\partial_x (aP) + \frac{1}{2}\partial_x^2 (b^2 P)}$$

Fokker-Planck equation

"Proof": Pick function $f(x)$.

$$\langle f \rangle(t) = \int dx \, f(x) \, P(x,t)$$ by definition.

$$\frac{d}{dt}\langle f \rangle = \int dx \, f(x) \, \partial_t P(x,t)$$ for any f.

Alternatively: $$P(x,t+dt) = \left\langle \int d\tilde{x} \, \delta(x - \tilde{x} - a(\tilde{x})dt - b(\tilde{x})d\xi) P(\tilde{x},t) \right\rangle_{d\xi}$$

here we use Itō

Here we are using infinitesimal time step forward in our ODE.

$$\langle f \rangle + \frac{d\langle f \rangle}{dt} dt = \int d\tilde{x} \left\langle f(\tilde{x} + a(\tilde{x})dt + b(\tilde{x})d\xi) \right\rangle_{\xi} P(\tilde{x},t)$$

$$= \int dx \left[f(x) + \partial_x f \cdot a \, dt + \partial_x f \cdot b \underbrace{\cancel{\langle d\xi \rangle}}_{0} + \frac{1}{2} \partial_x^2 f \cdot b^2 \underbrace{\cancel{\langle d\xi^2 \rangle}}_{dt} \right] P(x,t)$$

$$\frac{d\langle f \rangle}{dt} = \int dx \left(\partial_x f \cdot a + \frac{1}{2} b^2 \partial_x^2 f \right) P$$ for any f.

$$= \int dx \, f \left[-\partial_x(aP) + \frac{1}{2}\partial_x^2(b^2 P) \right]$$

And these two equations are only consistent if

$$\partial_t P = -\partial_x(aP) + \frac{1}{2}\partial_x^2(b^2 P)$$

which is the Fokker-Planck equation!

Why did we only Taylor expand to quadratic order?

Note: $[d\xi] = \sqrt{[\text{time}]}$, so $\langle d\xi^{2k} \rangle \sim dt^k$ negligible if $k>1$.

Non-Gaussian noise: $\langle d\xi^{2k} \rangle \sim \# \, dt + \cdots$

$\Rightarrow$ higher derivatives in FPE

We'll basically ignore non-Gaussian noise in this class.

The key point is that FPE is unambiguous, unlike SDEs.

Since we want effective theory to be unambiguous it makes sense to rely on FPE moving forward!

Let's now look at some examples.

Example 1: Brownian motion:

$$\frac{dx}{dt} = \sqrt{2D}\,\xi(t)$$ (the same as in Lecture 1)

$a = 0$, $b = \sqrt{2D}$:

$$\partial_t P = D\partial_x^2 P \rightarrow \text{diffusion equation.}$$

Assume particle starts at $x=0$, so $P(x,0) = \delta(x)$.

Then: $$P(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}.$$

$$= \int_{-\infty}^{\infty} dk\; e^{ikx}\,\tilde{P}(k,t) \quad \text{where} \quad \partial_t \tilde{P}(k,t) = -Dk^2\tilde{P}$$

or $\tilde{P}(k,t) = \tilde{P}(k,0)\,e^{-Dk^2t}$

where $$\tilde{P}(k,0) = \int_{-\infty}^{\infty} \frac{dx}{2\pi}\, e^{-ikx} P(x,0) = \frac{1}{2\pi}$$

$$\int_{-\infty}^{\infty} dk\; e^{ikx} \cdot \frac{1}{2\pi} e^{-Dk^2t} = \int_{-\infty}^{\infty} dk\; \frac{1}{2\pi}\, e^{-\frac{1}{2}\left(\sqrt{2Dt}\,k - \frac{ix}{\sqrt{2Dt}}\right)^2 - x^2/4Dt} = P(x,t)$$

Sanity check: $$\langle x(t)^2 \rangle = \int_{-\infty}^{\infty} dx\; x^2 P(x,t) = 2Dt$$

b/c the Gaussian distribution has mean 0 & variance $2Dt$.

Example 2: overdamped oscillator:

$$\cancel{m}\ddot{x}\to 0 \quad + \gamma \dot{x} + \mu x = \sigma\, \xi(t)$$

friction   Hooke's law   random forcing

$$\frac{dx}{dt} = -\frac{\mu}{\gamma}x + \frac{\sigma}{\gamma}\xi(t)$$

FPE

$$a = -\frac{\mu}{\gamma}x, \quad b = \frac{\sigma}{\gamma}$$

$$\partial_t P = \partial_x\left(\frac{\mu}{\gamma}x P\right) + \frac{\sigma^2}{2\gamma^2}\partial_x^2 P$$

Fourier transform as before:

$$\partial_t \tilde{P}(k,t) = -\frac{\sigma^2}{2\gamma^2}k^2\tilde{P} - \frac{\mu}{\gamma}k\partial_k \tilde{P}$$

Solve by method of characteristics:

$$\frac{dk}{dt} = \frac{\mu}{\gamma}k \Rightarrow k(t) = k(0)e^{\mu t/\gamma}$$

$$\frac{d\tilde{P}}{dt} = -\frac{\sigma^2}{2\gamma^2}k^2\tilde{P} = -\frac{\sigma^2 k(0)^2}{2\gamma^2}e^{2\mu t/\gamma}\tilde{P}$$

$$\log\frac{\tilde{P}(k(0),t)}{\tilde{P}(k(0),0)} = -\frac{\sigma^2}{2\gamma^2}k(0)^2\cdot\frac{\gamma}{2\mu}\left(e^{2\mu t/\gamma}-1\right) = -\frac{\sigma^2}{4\gamma\mu}k^2\left(1-e^{-2\mu t/\gamma}\right)$$

$k(0) = ke^{-\mu t/\gamma}$

Given $P(x,0) = \delta(x-y) \Rightarrow \tilde{P}(k,0) = e^{-iky}/2\pi$:

$$P(x,t) \sim \exp\left[-\frac{\gamma\mu}{\sigma^2(1-e^{-2\mu t/\gamma})}\left(x - ye^{-\mu t/\gamma}\right)^2\right]$$

On average: $\langle x(t)\rangle = \langle x(0)\rangle e^{-\mu t/\gamma}$

but the particle does not just diffuse away or come to rest.

Steady-state distribution:

$$\lim_{t\to\infty} P(x,t) = P_{ss}(x) = \# \cdot e^{-\Phi(x)} = \exp\left[-\frac{\gamma\mu}{\sigma^2}x^2\right]$$

independent of initial condition y

FPE is linear: overall prefactor uninteresting, fixes $\int dx\, P_{ss}(x) = 1$

So at late times the particle is trapped by combination of noise & restoring force.

If particle was in thermal equilibrium:

$$\Phi_{th}(x) = \beta H(x) = \frac{1}{T}\cdot\frac{\mu}{2}x^2$$

potential energy

Fluctuation - dissipation theorem: $\sigma^2 = 2T\gamma$

In lecture 3 we'll return to FDT and see that it is a manifestation of time-reversal symmetry in dissipative systems.

Stochastic equations with multiple DOF:

$$\dot{x}_i = a_i(\vec{x}) + b_{i\alpha}(\vec{x})\,\xi_\alpha(t) \qquad (\alpha = 1,\dots,m)$$

$(i = 1,\dots,n)$

$\langle \xi_\alpha(t)\rangle = 0$, $\langle \xi_\alpha(t)\xi_\beta(s)\rangle = \delta_{\alpha\beta}\,\delta(t-s)$

and we recall Einstein summation convention

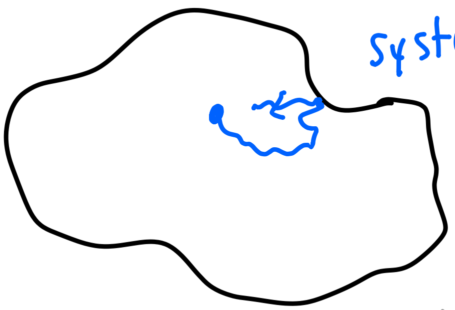
Repeating the same arguments from earlier in the class we get

Fokker-Planck equation for $P(x_1,\dots,x_n,t)$:

$$\partial_t P = -\partial_i(a_i P) + \frac{1}{2}\partial_i\partial_j(b_{i\alpha}b_{j\alpha}P)$$

Similar to Schrödinger equation, the PDE becomes higher-dimensional the more DOF we have.

Finite domains:



system cannot escape region R in configuration space.

What are the right boundary conditions? Notice that:

FPE: $\partial_t P + \partial_i J_i = 0$

$\hookrightarrow$ $J_i = a_i P - \frac{1}{2}\partial_j (b_{i\alpha} b_{j\alpha} P)$

J_i: probability current!

Particle can't escape $\Rightarrow$ $\hat{n}\cdot\vec{J} = 0$ at boundaries!

$\hookrightarrow$ for Brownian motion in interval $0 \le x \le L$:

FPE: $\partial_t P - D\partial_x^2 P = 0 \quad \Rightarrow \quad J = -D\partial_x P$

boundary conditions $\partial_x P(x=0, L) = 0.$

Solution to FPE:

$$P(x,t) = \frac{1}{L} + \sum_{n=1}^{\infty} c_n \cos\frac{n\pi x}{L}\, e^{-D\left(\frac{n\pi}{L}\right)^2 t}$$

$\frac{1}{L}$: uniform distribution, late-time steady-state

$$c_n = \frac{2}{L}\int_0^L dx \, \cos\frac{n\pi x}{L}\, P(x,0)$$

Solution method here analogous to particle in a box in QM so won't go through details.