PHYS 7810
Hydrodynamics
Spring 2026

Lecture 20
BBGKY hierarchy

March 26

So far we discussed hydrodynamics as EFT. But now turn to:

Kinetic theory: "microscopic derivation" of hydro (and beyond) for weakly interacting systems (e.g. gases)

This will also give us a more historical perspective on the subject.

Start with: classical Hamiltonian system with one type of well-defined particles, labeled $\alpha = 1, \ldots, N$ (↑ for simplicity)

Although this isn't most general ansatz, we'll assume that:

$$H = \sum_{\alpha=1}^{N} \left[\varepsilon(\vec{p}_\alpha) + U(\vec{x}_\alpha) \right] + \sum_{\alpha<\beta} V_2(\vec{x}_\alpha - \vec{x}_\beta)$$

($\varepsilon(\vec{p}_\alpha)$: single-particle kinetic energy; $U(\vec{x}_\alpha)$: external potential; V_2: two-body interactions)

Work in d spatial dimensions: (x_i, p_i) $i = 1, \ldots, d$ for each particle.

Poisson bracket: $\{x_{i\alpha}, p_{j\beta}\} = \delta_{ij}\delta_{\alpha\beta}$

collect into phase space $\xi^I_\alpha = (x_{i\alpha}, p_{i\alpha})$ "$I = 1, \ldots, 2d$"

Follow lectures 2-4 and consider:

Fokker-Planck (Liouville) equation for many-body probability distribution: $P(\xi, t)$:

$$\partial_t P = -\sum_{\alpha I} \frac{\partial}{\partial \xi^I_\alpha}\left(\{\xi^I_\alpha, H\} P\right)$$

But this equation is too high-dimensional to be useful. And most observables we care about are captured by

1-particle distribution: $f_1(\tilde{\xi}^I, t) = \int d^{2dN}\xi^I_\alpha \, P(\xi, t) \cdot \sum_{\alpha=1}^{N} \delta(\xi^I_\alpha - \tilde{\xi}^I)$

↗ number density of particles in phase space: write $f = f_1$

particle number density: $n(x,t) = \int d^d p \; f(x,p,t)$

momentum density: $g_i(x,t) = \int d^d p \; f(x,p,t) p_i$

Goal: find approximate equation for f, not P

It is useful to assume N particles are classically indistinguishable. In the FPE we can do this by imposing initial condition:

$P(\xi^I_1, \xi^I_2, \ldots) = P(\xi^I_2, \xi^I_1, \ldots)$ (permutation symmetric for all particles)

Since H is permutation symmetric: $P(\xi, t)$ stays symmetric.

$$f_1(\tilde{\xi}) = N \int d^{2d}\xi_2 \cdots d^{2d}\xi_N \, P(\tilde{\xi}, \xi_2, \ldots, \xi_N)$$

Generalize to n-particle correlations:

$$f_n(\tilde{\xi}_1, \ldots, \tilde{\xi}_n) = \frac{N!}{(N-n)!} \int d^{2d}\xi_{n+1} \cdots d^{2d}\xi_N \, P(\tilde{\xi}_1, \ldots, \tilde{\xi}_n, \xi_{n+1}, \ldots \xi_N)$$

Now let's calculate EOM for f_1:

$$\partial_t f_1 = \int d^{2d}\xi_2 \cdots d^{2d}\xi_N \, \partial_t P = -\int d^{2d}\xi_2 \cdots d^{2d}\xi_N \sum_{\alpha=1}^{N} \sum_{I=1}^{2d} \frac{\partial}{\partial \xi^I_\alpha}\left(\{\xi^I_\alpha, H\} P\right)$$

All $\frac{\partial}{\partial \xi^I_\alpha}$ terms are total derivatives except for $\alpha=1$. Therefore:

$$\partial_t f_1 = -N\sum_{I=1}^{2d} \frac{\partial}{\partial \xi_1^I} \int d^{2d}\xi_2 \cdots d^{2d}\xi_N \underbrace{\{\xi_1^I, H\}}P$$

$$\{x_{1i}, H\} = \frac{\partial H}{\partial p_{1i}} = v_i(p_1) \qquad \text{where} \quad v_i(p) = \frac{\partial \varepsilon}{\partial p_i}$$

$$\{p_{1i}, H\} = -\frac{\partial H}{\partial x_{1i}} = -\frac{\partial U}{\partial x}(x_1) - \sum_{\alpha=2}^{N} \frac{\partial}{\partial x_{1i}} V_2(\vec{x}_1 - \vec{x}_\alpha)$$

$$\partial_t f_1 = -N\int d^{2d}\xi_2 \cdots d^{2d}\xi_N \left[\frac{\partial}{\partial x_{1i}}\left(v_i(p_1)P\right) + \frac{\partial}{\partial p_{1i}}\left(-\frac{\partial U}{\partial x_{1i}}P - \sum_{\alpha=2}^{N} \frac{\partial V_2(\vec{x}_1 - \vec{x}_\alpha)}{\partial x_{1i}} P\right)\right]$$

Integrate over $\xi_2 \cdots \xi_N$ explicitly!

$$-\frac{\partial}{\partial x_{1i}}(v_i f_1) + \frac{\partial}{\partial p_{1i}}\left(\frac{\partial U}{\partial x_{1i}} f_1\right)$$

Integrate over $\xi_3 \cdots \xi_N$ (if $\alpha=2$)

$$+\frac{\partial}{\partial p_{1i}} \int d^{2d}\xi_2 \frac{\partial V_2(x_1 - x_2)}{\partial x_{1i}} f_2$$

So we arrive at an exact linear equation for f_1:

$$\underbrace{\partial_t f_1 + v_i \frac{\partial f_1}{\partial x_{1i}} - \frac{\partial U}{\partial x_{1i}} \frac{\partial f_1}{\partial p_{1i}}}_{\text{streaming terms}} = \underbrace{\int d^{2d}\xi_2 \frac{\partial V_2(x_1 - x_2)}{\partial x_{1i}} \frac{\partial f_2}{\partial p_{1i}}}_{\text{collision integral}}$$

BBGKY hierarchy comes from repeating these steps for f_n:

$$\underbrace{\partial_t f_n + \left\{f_n, \sum_{\alpha=1}^{n}(\varepsilon + U) + \sum_{\alpha<\beta=1}^{n} V_2\right\}}_{\text{streaming terms now involve first } n \text{ particles!}} = \int d^{2d}\xi_{n+1} \sum_{\alpha=1}^{n} \frac{\partial V_2(x_\alpha - x_{n+1})}{\partial x_{\alpha i}} \frac{\partial f_n}{\partial p_{\alpha i}}$$

How do we truncate these equations and get something more manageable?

Restrict to $n=1,2$: set $d=1$ to reduce clutter of indices;
and set $U=0$ for simplicity

$$\partial_t f_1 + v\frac{\partial f}{\partial x_1} = \int d\xi_2 \frac{\partial V_2(x_1-x_2)}{\partial x_1}\frac{\partial f_2}{\partial p_1}$$

$$\partial_t f_2 + v(p_1)\frac{\partial f_2}{\partial x_1} + v(p_2)\frac{\partial f_2}{\partial x_2} - \frac{\partial V_2(x_1-x_2)}{\partial x_1}\left(\frac{\partial f_2}{\partial p_1} - \frac{\partial f_2}{\partial p_2}\right)$$
$$= \int d\xi_3 \left[\frac{\partial V_2(x_1-x_3)}{\partial x_1}\frac{\partial f_3}{\partial p_1} + \frac{\partial V_2(x_2-x_3)}{\partial x_2}\frac{\partial f_3}{\partial p_2}\right]$$

Notice a separation of time scales between terms.

single particle: $\partial_t \sim \frac{1}{\tau_1} \sim \frac{\bar{v}}{l}$ ← typical velocity; l ← length scale of variation (l will be large!)

two particle: $\partial_t \sim \frac{1}{\tau_2} \sim \frac{V_2}{a}\cdot\frac{1}{\bar{p}}$ ← typical momentum; a ← range of interaction potential

collision: $\partial_t \sim \frac{1}{\tau_c} \sim \frac{V_2}{a}\cdot\frac{1}{\bar{p}}\cdot\int_{x_2:|x_2-x_1|\lesssim a} d\xi_2 f_2 \sim \frac{1}{\tau_2}\cdot n a^d$ ← particle density

Regime of kinetic theory $\frac{1}{\tau_1}\to\infty$, $na^d \ll 1$ so $\frac{1}{\tau_c} \ll \frac{1}{\tau_2}$.

So we can take a "steady state" approximation:

Ignore f_3 for simplicity, and set:

$$\frac{\partial V_2(x_1-x_2)}{\partial x_1}\left(\frac{\partial f_2}{\partial p_1} - \frac{\partial f_2}{\partial p_2}\right) \simeq v(p_1)\frac{\partial f}{\partial x_1} + v(p_2)\frac{\partial f}{\partial x_2}$$

Now switch to coordinates:

$$\bar{x} = \frac{x_1+x_2}{2} \qquad \tilde{x} = x_1 - x_2$$
$$\bar{p} = \frac{p_1+p_2}{2} \qquad \tilde{p} = p_1 - p_2$$

Assume $\frac{1}{\ell} \sim \frac{\partial}{\partial \tilde{x}} \ll \frac{\partial}{\partial \tilde{x}} \sim \frac{1}{a}$ in f_2:

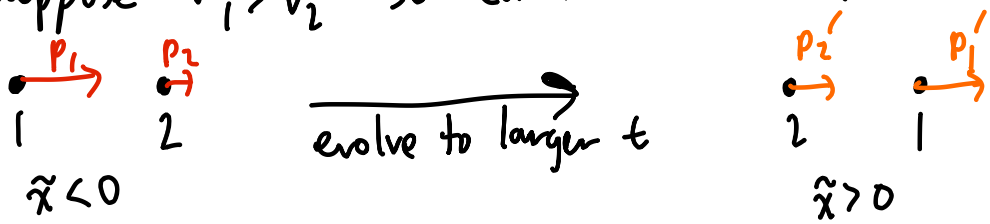
$$\frac{\partial V_2}{\partial \tilde{x}} \frac{\partial f_2}{\partial \tilde{p}} \approx \left[v(p_1) - v(p_2)\right] \frac{\partial f_2}{\partial \tilde{x}}$$

Plug in to the f_1 equation:

$$\partial_t f_1 + v(p_1) \frac{\partial f_1}{\partial x_1} \approx \int dp_2 \, d\tilde{x} \left[(v(p_1) - v(p_2)) \frac{\partial f_2}{\partial \tilde{x}} \right]$$

At this point one needs to start making some approximations and we'll just sketch how this is done.

Let's suppose $v_1 > v_2$ so collision occurs as:



w/ $v(p_1') > v(p_2')$.

If we integrate: $\int d\tilde{x} \frac{\partial f_2}{\partial \tilde{x}} = \underbrace{f_2(\tilde{x} \to \infty)}_{\text{after collision}} - \underbrace{f_2(\tilde{x} \to -\infty)}_{\text{before collision}}$

Assumption of molecular chaos:

$$\underbrace{f_2(\tilde{x} \to \infty)}_{\tilde{x} \gg a} \to f_1(p_1') f_2(p_2')$$

(particles uncorrelated after collision)

$$f_2(\tilde{x} \to -\infty) \to f_1(p_1) f_1(p_2)$$

difference is whether momenta are before or after collision...

One can rewrite the collision integral as:

$$\int dp_2 \, d\tilde{x} \left[v_1 - v_2\right] \frac{\partial f_2}{\partial \tilde{x}} \to \mathcal{C}[f_1] \quad \text{(collision integral)}$$ where

$$\mathcal{C}[f_1] = \mathcal{C}_{in}[f_1] - \mathcal{C}_{out}[f_1]$$

We'll express each part of the collision integral as

$$\mathcal{C}_{in}[f_1] = \int dp_2\, dp_1'\, dp_2' \; R(p_1' p_2' \to p_1 p_2)\, f_1(x, p_1')\, f_1(x, p_2')$$

$$\mathcal{C}_{out}[f_1] = \int dp_2\, dp_1'\, dp_2' \; R(p_1 p_2 \to p_1' p_2')\, f(x, p_1)\, f(x, p_2)$$

Since collision occurs on a short length scale, evaluate each f in collision integral at same point x.

$R(p_1' p_2' \to p_1 p_2)$ is scattering matrix element (e.g. from QFT) and usually has structure

$$R(p_1' p_2' \to p_1 p_2) = \delta(p_1' + p_2' - p_1 - p_2)\, \delta\big(\varepsilon(p_1') + \varepsilon(p_2') - \varepsilon(p_1) - \varepsilon(p_2)\big)\, |\mathcal{M}|^2$$

Putting this together we arrive at the Boltzmann equation:

$$f_1 \to f: \qquad \partial_t f + v_i(\vec{p}) \frac{\partial f}{\partial x_i} - \frac{\partial U}{\partial x_i} \frac{\partial f}{\partial p_i} = \mathcal{C}[f]$$

Here we generalized back to d dimensions & $U \neq 0$.

It's also possible to generalize the collision integral to more generic types of collisions – the most important cases deal w/ quantum mechanical particles:

bosons: $$\mathcal{C}[f] = \int dp_2\, dp_1'\, dp_2' \Big[R(p_1' p_2' \to p_1 p_2)\, f(p_1') f(p_2') (1 + f(p))(1 + f(p_2)) - R(p p_2 \to p_1' p_2')\, f(p) f(p_2) \underbrace{(1 + f(p_1'))(1 + f(p_2'))}_{\text{stimulated emission}} \Big]$$

fermions: $$\mathcal{C}[f] = \int dp_2\, dp_1'\, dp_2 \Big[R(p_1' p_2' \to p_1 p_2)\, f(p_1') f(p_2') (1 - f(p))(1 - f(p_2)) - R(p p_2 \to p_1' p_2')\, f(p) f(p_2) \underbrace{(1 - f(p_1'))(1 - f(p_2'))}_{\text{Pauli blocking}} \Big]$$