## PHYS 7810
## Hydrodynamics
## Spring 2026

# Lecture 21
## Boltzmann equation

March 31

Boltzmann equation: $$\partial_t f + \underbrace{\frac{\partial H_1}{\partial p_i}\frac{\partial f}{\partial x_i} - \frac{\partial H_1}{\partial x_i}\frac{\partial f}{\partial p_i}}_{\text{Streaming terms}} = \underbrace{\mathcal{C}[f]}_{\text{Collision integral}}$$

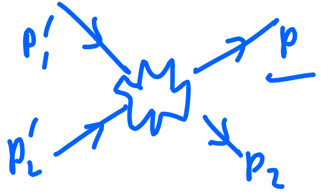
$f(x,p,t)$ = distribution function

$H_1$ = single-particle Hamiltonian $\to \varepsilon(p_i)$

In this class we won't derive the explicit form of the collision integral but the answer is of the form:

$$\mathcal{C}[f] = \int dp_2\, dp_1'\, dp_2' \left[ \underbrace{R(p_1' p_2' \to p\, p_2)} f(p_1') f(p_2') - \underbrace{R(p p_2 \to p_1' p_2')} f(p) f(p_2) \right]$$

incoming:



outgoing:

Energy and momentum conservation:

$$R(p p_2 \to p_1' p_2') = |\mathcal{M}|^2\, \delta(p + p_2 - p_1' - p_2')\, \delta(\varepsilon(p) + \varepsilon(p_2) - \varepsilon(p_1') - \varepsilon(p_2'))$$

scattering matrix element

Today's lecture is about the implications of Boltzmann equation. Let's first see how the molecular chaos assumption → collision integral leads to an arrow of time.

**Claim:** H-Theorem: $$-H = S = \int d^dx\, d^dp\, (-f\log f)$$

obeys $\boxed{\frac{dS}{dt} \geq 0} \rightarrow$ 2nd law of thermo.

Proof: $$\frac{dS}{dt} = \int d^dx\, d^dp \left(-\frac{\partial f}{\partial t}\left(\log f + \frac{f}{f}\right)\right) = -\int d^dx\, d^dp \left[\frac{\partial f}{\partial t}\log f + \cancel{\frac{\partial f}{\partial t}}\right]$$

0: particles conserved

(just for simplicity) assuming $H_1$ only depends on $p_i$

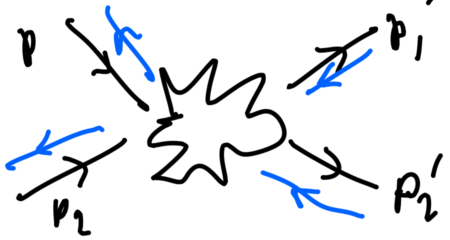
$$= -\int d^dx\, d^dp \left[\mathcal{C}[f] - \frac{\partial H_1}{\partial p_i}\frac{\partial f}{\partial x_i}\right]\log f$$

$$= -\int d^dx\, d^dp\, \mathcal{C}[f]\log f + \cancel{\int d^dx\, d^dp \frac{\partial}{\partial x_i}\left(\frac{\partial H_1}{\partial p_i}(f\log f - f)\right)}$$

=0 : total derivative

Now need to evaluate entropy produced by collisions:

Assume: time-reversal + inversion symmetry:



↳ $-(-\vec{p}) = \vec{p}$ unchanged

$$R(pp_2 \rightarrow p_1'p_2') = R(p_1'p_2' \rightarrow pp_2)$$

$$\frac{dS}{dt} = \int d^dx\, d^dp\, d^dp_2\, d^dp_1'\, d^dp_2'\; R(pp_2 \rightarrow p_1'p_2')\underbrace{[f(p)f(p_2) - f(p_1')f(p_2')]\log f(p)}$$

$$= \frac{1}{4}[f(p)f(p_2) - f(p_1')f(p_2')][\log f(p) + \log f(p_2) - \underbrace{\log f(p_1') - \log f(p_2')}]$$

to explain these terms, change integration variables $p \leftrightarrow p_1'$ and $p \leftrightarrow p_2'$

Note: $\log f(p) + \log f(p_2) = \log[f(p)f(p_2)]$.

And: $(X-Y)[\log X - \log Y] \geq 0$ b/c log monotonically increases

$\Rightarrow \frac{dS}{dt} \geq 0$ which proves the H-Theorem!

Next, let's show that the equilibria of the Boltzmann equation take a natural form:

Suppose $\frac{\partial f}{\partial t} = 0$. From H-Theorem: $\frac{dS}{dt} = 0$.

By the argument we just gave this is only possible if:

$f(p)f(p_2) = f(p_1')f(p_2')$ whenever $R(pp_2 \to p_1'p_2') \neq 0$, or:

$$\underbrace{\log f(p)} + \log f(p_2) = \log f(p_1') + \log f(p_2')$$

↳ $\log f(p)$ must be a conserved quantity in collisions!

$$\log f(p) = \underset{\uparrow \atop \text{particle number}}{A} + \underset{\uparrow \atop \text{momentum}}{B_i}\, p_i + \underset{\uparrow \atop \text{energy}}{C\,\varepsilon(p)}$$

Conventional to write as: $$f_{eq}(p) = \exp\left[-\frac{1}{\underset{\uparrow \atop \text{temperature}}{T}}\left(\varepsilon(p) - \underset{\uparrow \atop \text{chemical potential}}{\tilde{\mu}} - \underset{\uparrow \atop \text{fluid velocity}}{v_i}\, p_i\right)\right]$$

Maxwell-Boltzmann distribution for classical particles.

If we instead have quantum degrees of freedom:

fermions: $f_{eq}(p) = \frac{1}{e^{(\varepsilon - \cdots)/T} \underbrace{+1}}$ bosons: $f_{eq}(p) = \frac{1}{e^{(\varepsilon - \cdots)/T} \underbrace{-1}}$

In lecture 20 we discussed that collision integral was modified to:

$$C_{ferm} \sim R f f \underbrace{(1-f)(1-f)}_{\text{responsible for modified equilibrium}} \qquad C_{bos} \sim R f f \underbrace{(1+f)(1+f)}$$

The formula for entropy must also be modified:

fermion/boson : $S = \int d^dx\, d^dp \left[ -f\log f \mp (1\mp f)\log(1\mp f)\right]$

Now let's see how hydrodynamics arises. First let's indeed check that particles/energy/momentum are conserved!

Claim: $$\frac{d}{dt}\int d^dx\, d^dp\; f(x,p,t)\,\varepsilon(p) = 0$$

← or any other conserved quantity...

$$= \int d^dx\, d^dp\; \varepsilon(p)\left[ -\underbrace{\cancel{\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial x_i}}}_{\text{total der. in } x} + \mathcal{C}[f]\right]$$

Following same manipulations as before:

$$\int d^dp\; \varepsilon(p)\,\mathcal{C}[f] = \int d^dp\, d^dp_2\, d^dp_1'\, d^dp_2'\; \underbrace{\frac{\varepsilon(p)+\varepsilon(p_2)-\varepsilon(p_1')-\varepsilon(p_2')}{4}\, R(pp_2\to p_1'p_2')}_{=0:\ \text{energy conserved in collisions.}}\cdots$$

So we ought to be able to derive hydro from kinetic theory. Ideal hydro (w/o dissipation) can be elegantly derived!

Idea: on very long length scales,

$$f_{\text{hydro}}(x,p,t) \simeq \exp\left[-\frac{1}{T(x,t)}\left(\varepsilon(p) - \tilde{\mu}(x,t) - v_i(x,t)\,p_i\right)\right]$$

Hydro parameters are $x$-dependent, system in local equilibrium!

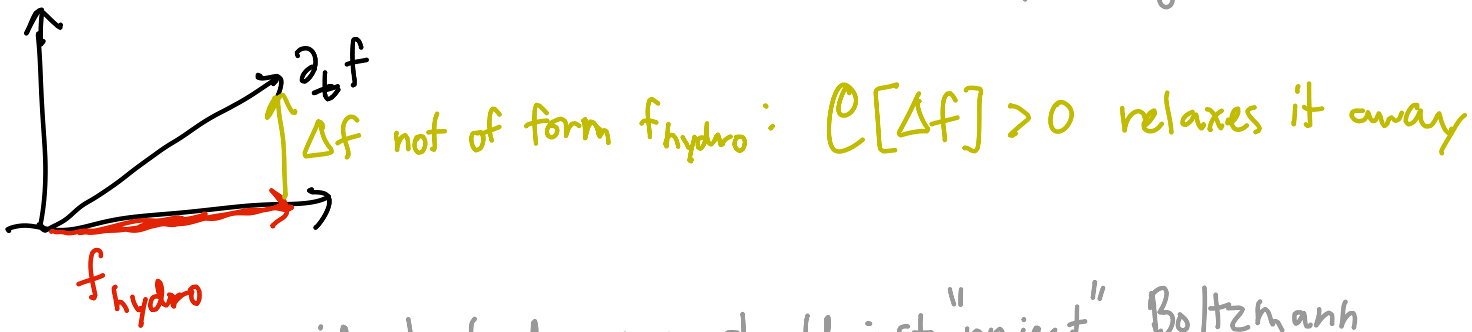
→ Plug-in: $\mathcal{C}[f_{\text{hydro}}] = 0$ by construction

Since $\mathcal{C}[f]$ is local in $x$.

Then: $$\partial_t f_{\text{hydro}} + \frac{\partial}{\partial x_i}\left(\boxed{\frac{\partial \varepsilon}{\partial p_i}}\, f_{\text{hydro}}\right) = 0\ ?$$

↳ Define as $V_i$ : single-particle velocity

More accurately, a sketch for what's happening is:



So to derive ideal hydro we should just "project" Boltzmann equation onto $f_{\text{hydro}}$ direction! (More in lecture 22!)

Idea: define conserved densities:

$$n(x) = \int d^d p\; f(x,p) \qquad g_i(x) = \int d^d p\; f(x,p)\, p_i \qquad \underset{\text{energy density}}{\varepsilon(x)} = \int d^d p\; f(x,p)\, \underset{\text{particle's kinetic energy}}{\varepsilon(p)}$$

Then: $\partial_t n + \int d^d p \left[ \frac{\partial}{\partial x_i} \left( V_i f_{\text{hydro}} \right) \right] = 0$

$\hookrightarrow$ $\partial_t n + \partial_i J_i = 0$ where current $J_i = \int d^d p\; f_{\text{hydro}} V_i$

Similarly: $\partial_t \varepsilon + \partial_i \mathcal{E}_i = 0$ where $\mathcal{E}_i = \int d^d p\; f_{\text{hydro}} V_i\, \varepsilon(p)$

$\partial_t g_i + \partial_j \tau_{ji} = 0$ where $\tau_{ji} = \int d^d p\; f_{\text{hydro}} V_j\, p_i$

We can now evaluate $n, g_i, \varepsilon, J_i, \mathcal{E}_i, \tau_{ji}$ in terms of $\tilde{\mu}, v_i, T$ to derive nonlinear ideal hydrodynamics!

Example: nonrelativistic ideal gas. $\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2m}$. $\beta = \frac{1}{T}$

$$n = \int d^d p\; e^{-\beta\left(\frac{p^2}{2m} - \tilde{\mu} - v_i p_i\right)} = \left(\frac{2\pi m}{\beta}\right)^{d/2} e^{\beta \underbrace{\left(\tilde{\mu} + \frac{m}{2} v^2\right)}_{\text{required by Galilean symmetry}}}$$

since $V_i = \frac{p_i}{m}$: $\underbrace{mJ_i = g_i}_{\text{symmetry!}} = \int d^d p\; e^{-\beta(\cdots)}\, p_i = m v_i \cdot n$

$$\tau_{ji} = \int d^d p \; \frac{p_j}{m} p_i \, e^{-\beta(\cdots)} = \int \underset{\nearrow}{d^d q} \; e^{\beta(\tilde{\mu} + m v^2/2)} \, e^{-\beta q^2/2m} \frac{(q_j + m v_j)(q_i + m v_i)}{m}$$

$$\vec{q} = \vec{p} - m\vec{v}$$

$$= m n v_i v_j + \delta_{ij} P \quad \text{where} \quad P = \frac{1}{m}\left(\frac{m}{\beta}\right) n = nT \leftarrow \text{ideal gas law!}$$

$$\varepsilon = \int d^d p \; \frac{p^2}{2m} e^{-\beta(\cdots)} = -\frac{1}{m\beta} \frac{\partial}{\partial\left(\frac{1}{m}\right)} n$$

$$= \frac{dn}{2\beta} - \frac{n}{m\beta}\left(\frac{\beta v^2}{2}\right)\left(-\frac{1}{\left(\frac{1}{m}\right)^2}\right) = \underbrace{\frac{dn}{2\beta}}_{\frac{d}{2}P \,=\, \text{internal}} + \underbrace{\frac{m n v^2}{2}}_{\text{overall kinetic}}$$

$$\mathcal{E}^0_i = \int d^d p \; \frac{p^2}{2m} \frac{p_i}{m} e^{-\beta(\cdots)} = \frac{1}{\beta m} \frac{\partial \varepsilon}{\partial v_i} = \frac{T}{2m}\left(dT + m v^2\right) \frac{\partial n}{\partial v_i} + n v_i T$$

$$= \frac{T}{2m} \cdot n \cdot \frac{m v_i}{T}\left(dT + m v^2\right) + n v_i T = \left(\frac{d}{2} + 1\right) n v_i T + \frac{n m v^2}{2} v_i$$

$$= (\varepsilon + P) v_i$$

We see that kinetic theory is fully consistent w/ the EFT constraints we derived on the Galilean-invariant fluid back in Lecture 8.