PHYS 7810
Hydrodynamics
Spring 2026

Lecture 22
Hydrodynamics from kinetic theory

April 2

Boltzmann equation: $$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} = C[f]$$
(without external forces)

Last time we argued that on long time scales we could approximate for a classical gas that

$$f \to \exp\left[-\beta(x,t)\left[\varepsilon(\vec{p}) - \tilde{\mu}(x,t) - v_i(x,t)\, p_i\right]\right] \text{ on large scales}$$

hydrodynamic degrees of freedom!

Today we will justify this more quantitatively by studying the linear response regime close to global thermal equilibrium:

Suppose: $$f(x,p,t) = \exp\left[-\beta\left(\varepsilon(\vec{p}) - \tilde{\mu}_0 - \Phi(x,p,t)\right)\right]$$
$$= f_{eq}(p) - \frac{\partial f_{eq}}{\partial \varepsilon}\Phi + \cdots \approx f_{eq}(1+\beta\Phi)$$

Keep only linear terms in Φ. Note we can make a similar ansatz for quantum Boltzmann equations using f_{eq} = Fermi-Dirac, e.g.

Plug into Boltzmann equation:

$$\partial_t(\beta f_{eq}\Phi) + v_i(p)\frac{\partial}{\partial x_i}(\beta f_{eq}\Phi) = C\left[f_{eq}(1+\beta\Phi)\right]$$

Recall: $\mathcal{C}[f] = \int d^dp_2\, d^dp_1'\, d^dp_2'\, R(pp_2 \to p_1'p_2')\left[f(p_1')f(p_2') - f(p)f(p_2)\right]$

$$\mathcal{C}[f_{eq}(1+\beta\Phi)] \approx \int d^dp_2\, d^dp_1'\, d^dp_2'\, R(pp_2 \to p_1'p_2')\, f_{eq}(p_1')f_{eq}(p_2') \times \beta\left[\Phi(p_1') + \Phi(p_2') - \Phi(p) - \Phi(p_2)\right]$$

[energy conservation: $f_{eq}(p_1')f_{eq}(p_2') = f_{eq}(p)f_{eq}(p_2)$]

So far this is a pretty messy integro-differential equation. But in the hydro regime we do know some modes should be slow:

$$\Phi(x,p,t) = \underbrace{\delta\tilde{\mu}(x,t) + \delta v_i(x,t)p_i - \frac{\delta\beta(x,t)}{\beta}\varepsilon(\vec{p})}_{\text{slow hydro modes?}} + \underbrace{\delta a(x,t)p_x p_y + \cdots}_{\text{fast non-hydro modes?}}$$

We have a linear equation for Φ; we could try to cleanly separate out the physics from our fast vs. slow modes. Let's try to build a mathematical framework which makes this precise.

Idea: $\int d^dp\, \Psi(p)\left[\beta f_{eq}\left(\partial_t\Phi + V_i\frac{\partial\Phi}{\partial x_i}\right) - \mathcal{C}\right] = 0$ for any Ψ:

Define inner product: $\langle\Psi|\Phi\rangle = \int d^dp\left(-\frac{\partial f_{eq}}{\partial\varepsilon}\right)\Psi\Phi = \int d^dp\ \beta f_{eq}\Psi\Phi$

Then: $\partial_t\langle\Psi|\Phi\rangle + \frac{\partial}{\partial x_i}\langle\Psi|\underbrace{V_i}_{}|\Phi\rangle + \underbrace{\langle\Psi|W|\Phi\rangle}_{} = 0$

matrix V_i acts as: $V_i|\Phi\rangle = |V_i\cdot\Phi\rangle$

W = linearized collision integral

Using formula above one can show that:

$$\langle\Psi|W|\Phi\rangle = \frac{1}{4}\int d^dp_1\, d^dp_2\, d^dp_1'\, d^dp_2'\ \beta f_{eq}(p_1)f_{eq}(p_2)R(p_1p_2\to p_1'p_2') \times \left[\Psi(p_1)+\Psi(p_2)-\Psi(p_1')-\Psi(p_2')\right]\left[\Phi(p_1)+\Phi(p_2)-\Phi(p_1')-\Phi(p_2')\right]$$

Note: w/ inversion + time-reversal, $W = W^T$

And W is positive semidefinite: $\langle \Phi | W | \Phi \rangle \geq 0$.

Since Boltzmann equation should hold for all $\langle \Psi |$ we have:

$$\left[\partial_t + V_i \frac{\partial}{\partial x_i} + W\right] | \Phi \rangle = 0 \quad : \quad \text{linearized Boltzmann equation}$$

Null vector of W $\Leftrightarrow$ conserved quantity in collision
$\Rightarrow$ hydrodynamic mode

since $\langle \Phi | W | \Phi \rangle = 0$ means $\Phi(p_1) + \Phi(p_2) = \Phi(p_1') + \Phi(p_2')$
when $R(p_1 p_2 \to p_1' p_2') > 0$.

Let's now find the quasinormal modes of the full Boltzmann equation.

$$\left[-i\omega + i k_i V_i + W\right] | \Phi \rangle = 0$$

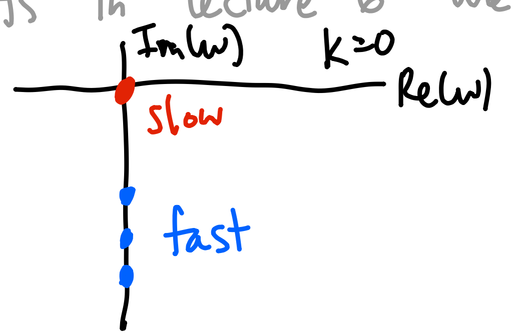
If $k_i = 0$: $\quad \omega | \Phi \rangle = -iW | \Phi \rangle$: QNMs = eigenvectors of W.
theory is stable: $\mathrm{Im}(\omega) \leq 0$. More precisely:

$$W = \left(\begin{array}{c|c} \begin{matrix} 0 & & \\ & 0 \ddots & \\ & & 0 \end{matrix} \;\; \text{hydro modes} & \text{(slow)} \quad 0 \\ \hline 0 & \begin{matrix} 1/\tau_1 & & \\ & 1/\tau_2 & \\ & & \ddots \end{matrix} \;\; \text{not hydro modes} \end{array}\right) \text{(fast)}$$

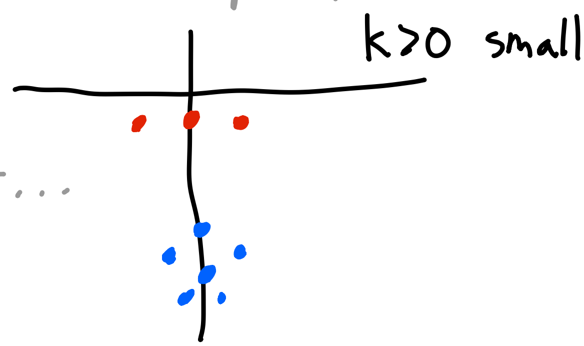
For a usual gas we have:

slow modes:	$\lvert n \rangle$	$\Phi = 1$	mass/number
	$\lvert p_i \rangle$	$\Phi = p_i$	momentum
	$\lvert \varepsilon \rangle$	$\Phi = \varepsilon(\vec{p})$	energy

As in lecture 6 we can plot QNMs in complex plane...:



and we might guess that...



We'd like to track what happens to the hydrodynamic poles.

Decompose: $|\Phi\rangle = \underset{\substack{\downarrow \\ \text{proportional to } |n\rangle, |\epsilon\rangle, |p_i\rangle}}{|\Phi_s\rangle} + \underset{\searrow \text{ everything else}}{|\Phi_f\rangle}$

$$-i\omega \begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix} + ik_i \begin{pmatrix} V_{ss,i} & V_{sf,i} \\ V_{fs,i} & V_{ff,i} \end{pmatrix} \begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & W_f \end{pmatrix} \begin{pmatrix} |\Phi_s\rangle \\ |\Phi_f\rangle \end{pmatrix} = 0.$$

If $\omega, V_{typ}k \ll \frac{1}{\tau}$ (eigenvalue of W): $\quad ik_i V_{fs,i}|\Phi_s\rangle + W_f|\Phi_f\rangle \approx 0$

Then: $$-i\omega|\Phi_s\rangle + ik_i V_{ss,i}|\Phi_s\rangle + \underbrace{ik_i V_{sf,i}|\Phi_f\rangle}_{} = 0$$

$$= \underbrace{k_i k_j V_{sf,i} W_f^{-1} V_{fs,j}}_{\text{effective collision integral } = W' \sim k^2} |\Phi_s\rangle$$

The k^2 scaling matches usual decay rates of hydro modes so we expect that we've now been able to capture viscosity!

Let's sketch how this might go in practice, focusing on the nonrelativistic gas. (You'll do a bit more on HW 6...):

Idea: $|\Phi_s\rangle = \delta\tilde{\mu}|n\rangle + \delta v_i|p_i\rangle + \frac{\delta T}{T}\left[|\epsilon\rangle - \tilde{\mu}_0|n\rangle\right]$

$$\begin{pmatrix} \langle n| \\ \langle p_i| \\ \langle \epsilon| \end{pmatrix} \left(-i\omega + ik_i V_{ss,i} + W'\right)|\Phi_s\rangle = 0.$$

To evaluate this expression we need to calculate a lot of inner products and matrix elements.

e.g. $$\langle \epsilon|\Phi_s\rangle = \int d^dp \left(-\frac{\partial f}{\partial \epsilon}\right)\Phi_s(p)\cdot \underbrace{\frac{p^2}{2m}}_{=\epsilon(p)} = \int d^dp\,(f - f_{eq})\frac{p^2}{2m} = \underset{\text{energy density}}{\delta\epsilon}$$

Similarly: $\langle n | \Phi_s \rangle = \delta n$ and $\langle p_i | \Phi_s \rangle = \delta g_i$

So eg. $-i\omega\, \delta n + i k_i \underbrace{\langle n | V_{ss,i} | \Phi_s \rangle}_{= \delta J_i \text{ (particle current)}} = 0$

Next we should deduce expressions for δn in terms of $\delta \mu$ etc...

$$\delta n = \langle n | \Phi_s \rangle = \delta\mu \langle n | n \rangle + \frac{\delta T}{T} \langle n | \varepsilon \rangle + \delta v_i \underbrace{\cancel{\langle n | p_i \rangle}}_{\to 0 \text{ by rotational symmetry}}$$

$$\langle n | n \rangle = \int d^d p\, \beta f_{eq} \cdot 1 \cdot 1 = \int d^d p\, \beta e^{\beta(\tilde{\mu} - p^2/2m)} = \left(\frac{2\pi m}{\beta}\right)^{d/2} \beta e^{\beta \tilde{\mu}} = \beta n_{eq}$$

$$\langle n | \varepsilon \rangle = \int d^d p\, \beta f_{eq} \cdot p^2/2m = \beta n_{eq} \cdot \frac{d}{2m} \cdot \frac{m}{\beta} = \frac{d}{2} n_{eq}$$

← so $|n\rangle, |\varepsilon\rangle$ not orthogonal!

$$\langle \varepsilon | \varepsilon \rangle = \int d^d p\, \beta f_{eq} \left(\frac{p^2}{2m}\right)^2 = \beta n_{eq} \cdot \frac{1}{4m^2}\left[d \cdot 3\left(\frac{m}{\beta}\right)^2 + d(d-1)\left(\frac{m}{\beta}\right)^2\right] = n_{eq} \cdot \frac{d(d+2)}{4\beta}$$

$$\langle p_i | p_j \rangle = m n_{eq} \delta_{ij}$$

In general: $\langle A | B \rangle = \chi_{AB} = \dfrac{\partial \langle B \rangle}{\partial \mu_A}$ (thermodynamic susceptibility)

$V_i = \dfrac{p_i}{m}$ so: $V_{ss,i} |n\rangle = \dfrac{1}{m} |p_i\rangle$ while $V_{fs,i} |n\rangle = 0$.

↳ $\langle n | (\partial_t + \cdots) | \Phi_s \rangle = 0 \to \partial_t \delta n + \partial_i (n_{eq} \delta v_i) = 0.$

$$V_i |p_j\rangle = \frac{1}{m} |p_i p_j\rangle = \frac{1}{m}\Big[\underbrace{|p_i p_j - \tfrac{1}{d}\delta_{ij} p^2\rangle}_{\text{fast non-hydro mode } (V_{fs})} + \underbrace{\frac{\delta_{ij}}{d} |p^2\rangle}_{\text{slow hydro mode } (V_{ss})} \Big], \quad |p^2\rangle = 2m|\varepsilon\rangle$$

Since $V_{ss,i} |p_j\rangle = \dfrac{2m}{d} \delta_{ij} |\varepsilon\rangle$: $\; i k_i \langle p_j | V_{ss,i} | \Phi_s \rangle = i k_j (m n \delta\mu + s \delta T) = i k_j \overbrace{\delta P}^{\text{pressure}}$

where s = entropy density

As expected the ideal hydro matches Navier-Stokes, as we already saw in lecture 21. The key is that we can now also handle dissipative contributions as well:

$$\langle p_\ell | W' | \Phi_s \rangle = \langle p_\ell | V_{sf,i} W_f^{-1} V_{fs,j} k_i k_j | \Phi_s \rangle$$
$$= \frac{k_i k_j \delta v_k}{m^2} \underbrace{\langle p_i p_\ell - \tfrac{p^2}{d}\delta_{i\ell} | W_f^{-1} | p_j p_k - \tfrac{p^2}{d}\delta_{jk} \rangle}$$

use rotational symmetry to simplify? $= \# \left[\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} - \frac{2}{d}\delta_{i\ell}\delta_{jk}\right]$

$$\delta_{ij}\delta_{k\ell}\left[\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} - \frac{2}{d}\delta_{i\ell}\delta_{jk}\right] = d^2 + \left(1-\frac{2}{d}\right)d = d^2 + d - 2$$

This trick will be helpful on HW6.

Take $i=j=x$, $k=\ell=y$: $\quad k_x^2 \delta v_y \underbrace{\frac{\langle p_x p_y | W_f^{-1} | p_x p_y \rangle}{m^2}}_{=\eta \text{ (shear viscosity)}}$

More generally: dissipative coefficients from:

$$\langle \underbrace{\Phi_s | V_{sf,i}} W_f^{-1} \underbrace{V_{fs,j} | \Phi_s} \rangle$$

part of conserved current not overlapping a fast mode.

If: $\eta_{ijk\ell} = \frac{1}{m^2}\langle p_i p_j - \frac{p^2}{d}\delta_{ij} | W_f^{-1} | p_k p_\ell - \frac{p^2}{d}\delta_{k\ell}\rangle$

$\Rightarrow$ bulk viscosity $\zeta = 0$: $|p_i p_j - \frac{p^2}{d}\delta_{ij}\rangle$ has no traceful part

It's very hard to actually evaluate W_f^{-1}, so a simple approximation:

Relaxation time approximation: $W_f^{-1}|\Phi_f\rangle = \tau |\Phi_f\rangle$

$\tau \leftarrow$ mean free time = constant!

Then: $\eta = \frac{\langle p_x p_y | p_x p_y\rangle \tau}{m^2} = \frac{\tau}{m^2}\beta n_{eq}\left(\frac{m}{\beta}\right)^2 = \tau n_{eq} T = \tau P \leftarrow$ pressure

As $\tau \to \infty$, $\eta \to \infty$

So very viscous fluids often have very weak interactions!
In Lecture 23 we will see more concretely why this should be the case within kinetic theory.