PHYS 7810
Hydrodynamics
Spring 2026

Lecture 23
Ballistic-to-viscous crossover

April 7

Linearized Boltzmann equation: $\left[-i\omega + ik_iV_i + W\right]|\Phi\rangle = 0.$

streaming: $V_i = \frac{\partial \varepsilon}{\partial p_i}$

linearized collision integral

$f = f_{eq} - \frac{\partial f_{eq}}{\partial \varepsilon}\Phi + \cdots$

Inner product: $\langle \Phi | \Psi \rangle = \int d^dp \left(-\frac{\partial f_{eq}}{\partial \varepsilon}\right) \Phi \Psi.$

Last time we projected Φ onto slow (hydro) vs. fast (non-hydro) modes:

Project onto slow modes: P_s projects onto $\mathrm{span}\{|n\rangle, |\varepsilon\rangle, |p_i\rangle\}$

$$P_s\left[-i\omega + ik_iV_i + W\right]|\Phi\rangle = 0$$

$$\hookrightarrow \left[-i\omega + \underbrace{ik_iV_{ss,i}}_{\text{ideal hydro/thermodynamic}} + \underbrace{k_iV_{sf,i}W_f^{-1}V_{fs,j}k_j}_{\text{dissipative corrections}}\right]P_s|\Phi\rangle = 0$$

Problem: how do we actually evaluate W_f^{-1}? In lec. 22 we used a relaxation time approximation but we also have microscopic formulas for collision integrals we'd like to use.

Prop: Chapman-Enskog variational principle:

$$\langle \Psi | M^{-1} | \Psi \rangle = \max_{|\Phi\rangle} \frac{\langle \Phi | \Psi \rangle^2}{\langle \Phi | M | \Phi \rangle} \rightarrow \text{call this } \mathcal{R}.$$

Proof: assume finite-dimensional basis $|\Phi\rangle = \sum \Phi_\alpha |\alpha\rangle$ for simplicity

$$\frac{\partial \mathcal{R}}{\partial \Phi_\alpha} = 2\frac{\Psi_\alpha \langle \Phi | \Psi \rangle}{\langle \Phi | M | \Phi \rangle} - \frac{\langle \Phi | \Psi \rangle^2}{\langle \Phi | M | \Phi \rangle^2} 2 M_{\alpha\beta} \Phi_\beta = 0 \quad \text{when}$$

normalization not important: $\mathcal{R}[|\Phi\rangle] = \mathcal{R}[\lambda|\Phi\rangle]$

$$\Psi_\alpha \sim M_{\alpha\beta}\Phi_\beta \quad \text{or} \quad \Phi_\alpha = (M^{-1})_{\alpha\beta}\Psi_\beta$$

$$\mathcal{R} = \frac{\langle \Psi | M^{-1} | \Psi \rangle^2}{\langle \Psi | M^{-1} M M^{-1} | \Psi \rangle} = \langle \Psi | M^{-1} | \Psi \rangle. \quad \blacksquare$$

You'll get practice using this on HW6. The basic idea in practice is usually to just guess...

Ansatz: $|\Phi\rangle = \sum_{\alpha=1}^{M} \Phi_\alpha |\alpha\rangle$ where $P_s|\alpha\rangle = 0$. ← because V_{fs} projected out slow modes.

guess M basis vectors (NB: don't include terms whose inner product w/ $|\Psi\rangle$ is zero by symmetry)

Example 1: shear viscosity. Assume rotational symmetry

Momentum: $-i\omega |p_i\rangle \underbrace{\langle p_i | \Phi\rangle}_{\sim v_i} + \cdots + \underbrace{|p_i\rangle\langle p_i| k_k V_{sf,k} W_f^{-1} V_{fs,\ell} k_\ell |p_j\rangle}_{\text{contains } \eta \nabla^2 v \ldots} \underbrace{\langle p_j | \Phi\rangle}_{\sim v_j} = 0$

$|\tau_{xy}\rangle = |V_x p_y\rangle$

Claim: $$\eta \geq \frac{\langle \tau_{xy} | \tau_{xy} \rangle}{\langle \tau_{xy} | W | \tau_{xy} \rangle}$$ ← plug in guess $|\Phi\rangle = |\tau_{xy}\rangle$ into variational method

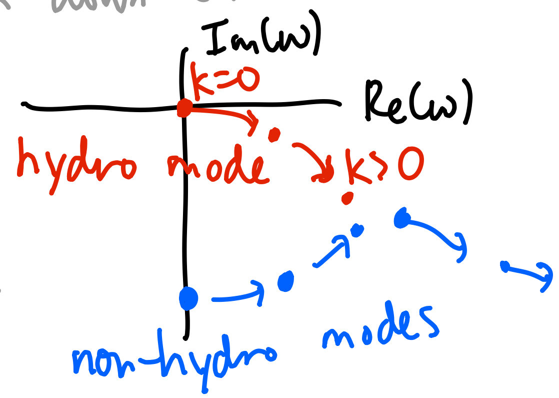
By symmetry: $|\tau_{xy}\rangle$ is spin-2 → no overlap w/ conserved Q's. (in ordinary gas)

More generally, dissipative hydro coefficients come from matrix elements of currents. Remember to project out overlap w/ any conserved quantities!

Recall that hydro is an EFT valid on long time scales, but it can break down on shorter time scales...

Schematic:

Poles in correlation functions at...



} all in kinetic theory!

Expectation: if $\frac{1}{\tau}$ is mean free time (smallest W_f eigenvalue):

$|\omega| \ll \frac{1}{\tau}$: hydrodynamic

$|\omega| \gg 1/\tau$: ballistic (not hydro)

$\underbrace{\hphantom{|\omega| \gg 1/\tau : \text{ballistic (not hydro)}}}_{\text{approximately \underline{single-particles}}}$

We can use kinetic theory to quantitatively describe this crossover in a toy model.

Example 2: gas of massless relativistic particles $(d=1)$:

$$\varepsilon(p) = c|p| \rightsquigarrow V = c\cdot \mathrm{sign}(p)$$

Hydro modes: $|n\rangle$ $(\Phi=1)$ $\quad |p\rangle$ $(\Phi=p)$ $\quad |\varepsilon\rangle$ $(\Phi=c|p|)$ } P_s = project onto these modes

Relaxation time approx: $W = \gamma(1-P_s)$

Boltzmann: $[-i\omega + ikV + \gamma(1-P_s)]|\Phi\rangle = 0.$

Following lecture 6 we can calculate G^R:

Claim: if $A = \int d^d p \; a(p) f(p)$, $B = \int d^d p \; b(p) f(p)$

$\rightarrow \langle a | \Phi \rangle$ $\rightarrow \langle b | \Phi \rangle$

Then $G^R_{AB}(k,\omega) = \langle a | (-i\omega + ik_i V_i + W)^{-1} | b \rangle$

For our particular problem we have:

$$-i\omega + ik_i V_i + W = \underbrace{[-i\omega + \gamma + ik_i V_i]}_{\text{call } G_0^{-1}\text{, where } G_0(p,p') = \frac{\delta(p-p')}{\gamma - i\omega + ikc\,\mathrm{sign}(p)}} - \gamma P_s$$

Evaluate: $G = (-i\omega + ik_i V_i + W)^{-1} = (G_0^{-1} - \gamma P_s)^{-1}$

$$= G_0 + G_0(\gamma P_s) G_0 + G_0(\gamma P_s G_0 \gamma P_s) G_0 + \cdots$$

$$= G_0 + G_0 P_s \gamma(1-\gamma\tilde{G}) P_s G_0 \quad \text{where } \tilde{G} = P_s G_0 P_s.$$

This trick allows us to formally solve this relaxation time problem. In particular let's look at the density correlation function:

$$\langle n | G | n \rangle = G^R_{nn} = \langle n | \tilde{G} + \tilde{G}\gamma(1-\gamma\tilde{G})\tilde{G} | n \rangle = \langle n | \tilde{G}(1-\gamma\tilde{G})^{-1} | n \rangle.$$

To evaluate $\tilde{G}$ we need to find an orthonormal basis for slow modes.

$$|1\rangle = \frac{|n\rangle}{\sqrt{\langle n|n\rangle}} \qquad |2\rangle = \frac{|p\rangle}{\sqrt{\langle p|p\rangle}} \qquad |3\rangle = \frac{|\tilde{\varepsilon}\rangle}{\sqrt{\langle\tilde{\varepsilon}|\tilde{\varepsilon}\rangle}}, \quad |\tilde{\varepsilon}\rangle = |\varepsilon\rangle - |n\rangle \frac{\langle n|\varepsilon\rangle}{\langle n|n\rangle}$$

$$\langle n|n\rangle = \int dp \left(-\frac{\partial f_{eq}}{\partial \varepsilon}\right) \cdot 1 = \int dp \; \beta e^{-\beta c|p|} = \frac{2}{c}$$

$$\langle p|p\rangle = \int dp \; \beta e^{-\beta c|p|} p^2 = \frac{4}{\beta^2 c^3} = \frac{\langle\varepsilon|\varepsilon\rangle}{c^2}$$

$$\langle n|\varepsilon\rangle = \int dp \; \beta e^{-\beta c|p|} c|p| = 2/\beta c$$

So $\langle \tilde{\varepsilon} | \tilde{\varepsilon} \rangle = \frac{4}{\beta^2 c} - \left(\frac{2}{\beta c}\right)^2 \frac{c}{2} = \frac{2}{\beta^2 c}$

$$\langle 1 | \tilde{G} | 1 \rangle = \frac{c}{2} \int_{-\infty}^{\infty} dp \; \beta e^{-\beta c |p|} \cdot \frac{1}{\gamma - i\omega + ikc \cdot \mathrm{sign}(p)}$$

$$= \frac{\beta c}{2} \cdot \left[\frac{1}{\beta c}\right] \left[\frac{1}{\gamma - i\omega + ikc} + \frac{1}{\gamma - i\omega - ikc}\right] = \frac{\gamma - i\omega}{(\gamma - i\omega)^2 + (ck)^2}$$

After some more calculations:

$$\tilde{G} = \frac{1}{(\gamma - i\omega)^2 + (ck)^2} \begin{array}{c} \begin{array}{ccc} |1\rangle & |2\rangle & |3\rangle \end{array} \\ \begin{pmatrix} \gamma - i\omega & -ick/\sqrt{2} & 0 \\ -\frac{ick}{\sqrt{2}} & \gamma - i\omega & -ick/\sqrt{2} \\ 0 & -\frac{ick}{\sqrt{2}} & \gamma - i\omega \end{pmatrix} \end{array}$$

$$\hookrightarrow G^R_{nn} = \langle 1 | \tilde{G}(1 - \gamma \tilde{G}) | 1 \rangle \cdot \langle n | n \rangle = \frac{2}{c} \frac{-2i\omega(c^2k^2 - \omega^2) + \gamma(c^2k^2 - 2\omega^2)}{2\underbrace{(c^2k^2 - \omega^2)}\underbrace{(c^2k^2 - \omega^2 - i\omega\gamma)}}$$

Poles $\rightarrow$ excitations: hydrodynamic sound modes: $\omega = \pm ck$

$$\omega = -i \, \frac{\gamma \pm \sqrt{\gamma^2 - (2ck)^2}}{2} \longrightarrow \begin{cases} \boxed{-i \frac{c^2}{\gamma^2} k^2} \leftarrow \text{hydro} \\ \boxed{-i\gamma} \end{cases} \text{ as } k \to 0$$

non-hydro

Why does sound mode not decay?

$V|\varepsilon\rangle = c^2 |p\rangle$ and $V|p\rangle = |\varepsilon\rangle$ $\qquad 1 - \frac{\beta}{2}\varepsilon$

$\hookrightarrow$ perfect decoupling! $\quad \partial_t \varepsilon + c^2 \partial_x g = 0$

$\qquad \partial_t g + \partial_x \varepsilon = 0$

We have incoherent diffusion for density mode:

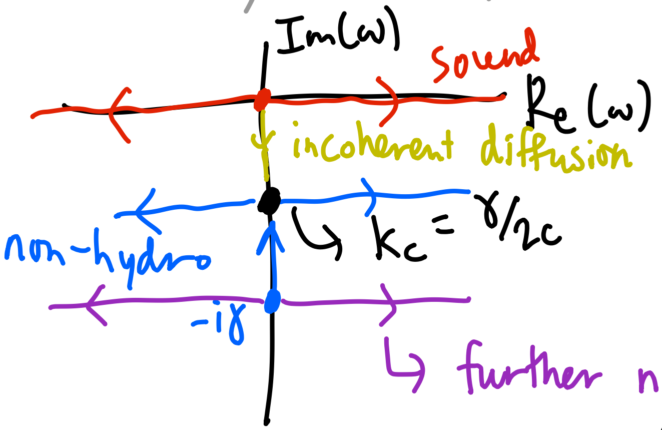
$|n\rangle$ overlaps w/ $|\varepsilon\rangle$ so the current

$|J\rangle = |c \cdot \mathrm{sign}(p)\rangle$ $\underbrace{\text{overlaps w/ } |p\rangle}_{\text{hydro (sound) mode}}$ & $\underbrace{|\mathrm{sign}(p) - \tfrac{1}{2}\beta c p\rangle}_{\text{non-hydro mode}}$

$\downarrow$

leads to incoherent charge diffusion

In summary we detect the following QNMs from kinetic theory:



where arrows denote what happens as k increases

↳ further non-hydro modes: $\omega = \pm ck - i\gamma$

since $V^2 = c^2$ exchanges 2 non-hydro modes:

$$\partial_t \phi_1 + c\,\partial_x \phi_2 + \gamma \phi_1 = 0 \qquad \text{w/ } V|\phi_1\rangle = c|\phi_2\rangle$$
$$\partial_t \phi_2 + c\,\partial_x \phi_1 + \gamma \phi_2 = 0 \qquad V|\phi_2\rangle = c|\phi_1\rangle$$

Higher dimensions: $\tau_{ij} = |V_i p_j\rangle$ has overlap w/ fast modes

$\Rightarrow$ shear viscosity $\eta > 0$ & sound decays.