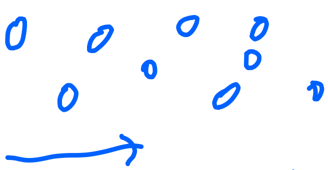
PHYS 7810
Hydrodynamics
Spring 2026

# Lecture 24
## Superfluids

April 9

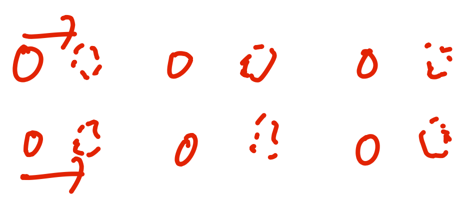
Today we'll discuss hydrodynamic EFTs w/
Spontaneous symmetry breaking (SSB): state not invariant under symmetry of EOMs.

Liquid:



looks same under translation

Solid:



SSB of translation

SSB leads to distinct phases of matter, and hydrodynamic EFTs are qualitatively modified by SSB.

Today we'll talk about a minimal model of SSB:

U(1) symmetry: global conserved (integer) charge $Q$.

particle number? $\rho = \bar{\psi}\psi$, $Q = \int d^dx\, \rho$.

(Nonlinear) Schrödinger equation: $i\partial_t \psi = -\frac{1}{2m}\partial_i\partial_i\psi + \tilde{\mu}(|\psi|^2)\psi$

U(1): phase rotation $\psi(x,t) \to e^{i\phi_0}\psi(x,t)$ leaves EOM inv.

Lagrangian: $\mathcal{L} = i\bar{\psi}\partial_t\psi - \frac{1}{2m}\partial_i\bar{\psi}\partial_i\psi - \varepsilon(\bar{\psi}\psi)$ $\hookrightarrow \tilde{\mu} = V'(\bar{\psi}\psi)$

invariant under $\psi \to e^{i\phi_0}\psi$ & $\bar{\psi} \to e^{-i\phi_0}\bar{\psi}$

If continuous symmetries are associated w/ conserved densities then what if we try to write $\mathcal{L}$ in terms of hydro variables?

Write $\psi = \sqrt{\rho}\, e^{i\phi}$, $\bar{\psi} = \sqrt{\rho}\, e^{-i\phi}$; U(1) is $\phi \to \phi + \phi_0$ (nonlinearly realized)

After some manipulations:

$$\mathcal{L} \to -\rho\partial_t\phi - \frac{\rho}{2m}\partial_i\phi\partial_i\phi - \frac{1}{8m\rho}\partial_i\rho\partial_i\rho - \varepsilon(\rho)$$

Hamiltonian mechanics: $\{\rho(x), \phi(x')\} = \delta(x-x')$

$$H = \varepsilon(\rho) + \frac{K}{2}\partial_i\phi\partial_i\phi + \cdots \quad \text{where } K = \frac{\rho}{m}$$

most general consistent w/ U(1) shift

So from this microscopic perspective we should have introduced a conjugate field $\phi$ to $\rho$ when building hydro EFT??

U(1) unbroken $\longrightarrow$ normal fluid: $H = \varepsilon(\rho) + \cdots$

invariant under enhanced $\phi \to \phi + f(x)$ shift.

$\partial_t\phi = -\frac{\partial\varepsilon}{\partial\rho} = -\tilde{\mu}(\rho)$ is a redundant DOF.

And the dynamics not visible until we extend to MSR dissipative EFT

U(1) SSB $\longrightarrow$ superfluid: $H = \frac{K}{2}(\nabla\phi)^2 + \varepsilon(\rho) + \cdots$

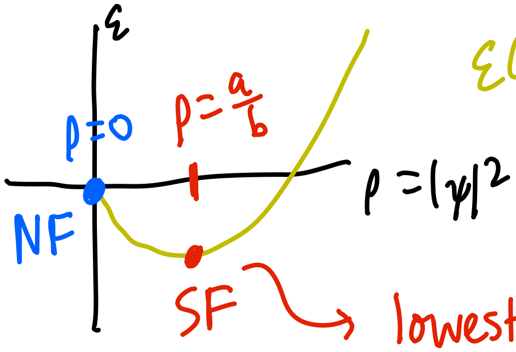
only has global $\phi \to \phi + \phi_0$ symmetry.

$\partial_t\phi = -\tilde{\mu}$ AND $\partial_t\rho = -K\nabla^2\phi$

How do these 2 phases arise in our minimal model?

Superfluid has $K \neq 0$ so: $\frac{\rho}{m} \neq 0$ or $\rho \neq 0$.

A useful cartoon is to consider:




$$\varepsilon(\rho) = -a\rho + \frac{b}{2}\rho^2$$

lowest energy state is a SF

In this model if $\rho \neq 0$ we have SF, but in nature it is certainly possible to have $\rho \neq 0$ but not see any SF behavior. We'll return to this discussion more in lecture 25.

In a superfluid the relative phase is physical so hydro includes:

$\phi =$ Goldstone boson, a new hydro DOF for superfluid.

Now let's develop MSR Lagrangian for dissipative SF hydro:

Steady state $\Phi = \beta H = \beta \int d^d x \left( \frac{K}{2}(\nabla\phi)^2 + \varepsilon(\rho) + \cdots \right)$

$$\mu_\rho = \frac{\delta \Phi}{\delta \rho} = \beta \tilde{\mu} \quad \text{and} \quad \mu_\phi = \frac{\delta \Phi}{\delta \phi} = -\beta K \nabla^2 \phi$$

$$\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_\phi \partial_t \phi - \overset{1/\beta}{T}\left[\pi_\rho \mu_\phi - \pi_\phi \mu_\rho\right] + iT\sigma \partial_i \pi_\rho \partial_i (\pi_\rho - i\mu_\rho)$$
$$+ iT\alpha\, \pi_\phi(\pi_\phi - i\mu_\phi) + \cdots$$

encodes Poisson brackets

EOMs: $\partial_t \rho + K \partial_i \partial_i \phi - \partial_i\left(\frac{\sigma}{\chi}\partial_i \rho\right) = 0$ if $\tilde{\mu}(\rho) \approx \frac{\rho}{\chi} + \cdots$

$$\partial_t \phi + \tilde{\mu} - \alpha K \partial_i \partial_i \phi = 0$$

Ideal hydro limit (no dissipation, $\sigma = \alpha = 0$):

$$\partial_t^2 \phi = -\partial_t \tilde{\mu} = -\frac{1}{\chi}(-K\partial_i\partial_i\phi) = v_s^2 \nabla^2\phi \quad \text{where} \quad v_s = \sqrt{\frac{K}{\chi}}$$

$\Rightarrow$ dispersion relation $\omega = \pm v_s k$: superfluid sound modes.

Indeed a natural picture for this phenomenon is that at ideal hydro level we could integrate out $\rho$:

$$\mathcal{L}_{\text{ideal}} = -\rho\partial_t\phi - \frac{K}{2}\partial_i\phi\partial_i\phi - \frac{\rho^2}{2\chi} \longrightarrow \underbrace{\mathcal{L} = \frac{\chi}{2}(\partial_t\phi)^2 - \frac{K}{2}\partial_i\phi\partial_i\phi}_{\text{effective Lagrangian invariant under } U(1):\ \phi \to \phi + \phi_0}$$

So symmetry enforces Goldstone boson to be a "massless" particle!

Now let's put dissipation back in to the superfluid EFT.

$$-i\omega\begin{pmatrix}\rho \\ \phi\end{pmatrix} = \begin{pmatrix} -\frac{\sigma}{\chi}k^2 & Kk^2 \\ -\frac{1}{\chi} & -\alpha K k^2\end{pmatrix}\begin{pmatrix}\rho \\ \phi\end{pmatrix}$$

$$\hookrightarrow \omega = \pm v_s k - \frac{i}{2}k^2\left[\frac{\sigma}{\chi} + \alpha K\right] + \cdots$$

Limit of normal fluid: $K \to 0$ and $\omega = -i\frac{\sigma}{\chi}k^2$

Hence we reproduce the EFT of Lecture 5 upon restoring the $\phi \to \phi + f(x)$ symmetry of a normal fluid.

Thus far our discussion has assumed that superfluids exist. But:

Mermin-Wagner Theorem: finite T SF impossible in $d \leq 2$.

If we did have a SF phase we'd want to have:

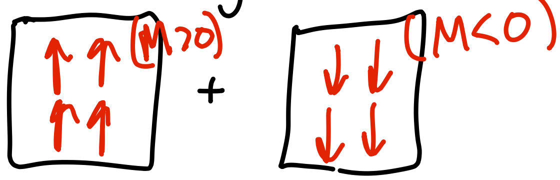
$$\lim_{|x-x'|\to\infty} \langle e^{i\phi(x)} e^{-i\phi(x')}\rangle > 0$$

This defines long-range order in charged observable

$$e^{i\phi(x)} \to e^{i\phi_0} e^{i\phi(x)} \quad \text{under } U(1)$$

Analogy: ordered ferromagnet:

$$\langle \cdots \rangle \sim$$




Configurations related by symmetry.

magnetization

So: $\langle M \rangle \sim M_0 + (-M_0) = 0$

but: $\langle M(x)\, M(y) \rangle \sim M_0 \cdot M_0 + (-M_0)\cdot(-M_0) > 0$

Suppose we indeed had long-range order in SF.
Calculate thermal correlation function by doing path integral:

$$\langle e^{i\phi(x)} e^{i\phi(x')} \rangle \sim 1 - \langle \phi(x)\phi(x') \rangle + \cdots$$

$$\sim 1 - \int D\phi \;\; \phi(x)\,\phi(x')\, e^{-\beta \int d^d x \; \frac{K}{2}(\nabla\phi)^2}$$

$$\sim 1 - \frac{1}{\beta K} \underbrace{(-\nabla^2)^{-1}(x,x')}$$

Green's function $G(x-x') \sim \begin{cases} r & d=1 \\ \log r & d=2 \\ \frac{1}{r^{d-2}} & d>2 \end{cases}$ where $r = |x-x'|$

universal part of correlator grows as $r\to\infty$

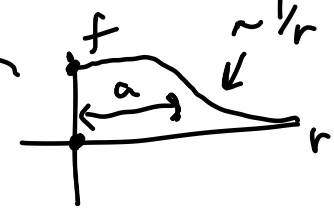
universal part decays as $r \to \infty$, consistent w/ SF

Thus we must have $d>2$ to see SF phase and hydrodynamics!

Note: SF phase often disappears for $T > T_c$ as $K \to 0$.

<u>$d=2$:</u> SF destroyed by vortex formation:

$$\psi \sim \sqrt{\rho}\, e^{i\phi} \sim (x+iy) \cdot f\left(\sqrt{x^2+y^2}\right) ?$$




or $\quad \phi \sim \arctan\left(\frac{y}{x}\right)$

Energy cost of isolated vortex: $E \sim \int d^2x\, \frac{K}{2} (\nabla\phi)^2 \sim \int 2\pi r\, dr \cdot \frac{K}{2r^2}$

$$\sim \pi K \log \frac{L}{a}$$

$L$ ← system size

$a$ ← vortex core size

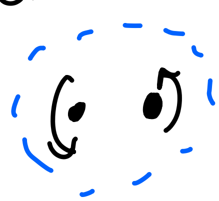
Thermal entropy: vortex can be placed in $\left(\frac{L}{a}\right)^2$ places?

$$S \sim \log\left(\frac{L}{a}\right)^2.$$

Minimize free energy: $F = E - TS = (\underbrace{\pi K - 2T}_{T_c = \frac{\pi}{2} K}) \log \frac{L}{a}$

For $T > T_c$, vortices proliferate

$T < T_c$, vortices form bound pairs which proliferate:



↳ leads to quasi-long-range order:

$$\langle e^{i\phi(x)} e^{-i\phi(x')} \rangle \sim \frac{1}{|x-x'|^a}$$