PHYS 7810
Hydrodynamics
Spring 2026

Lecture 26
Strong-to-weak spontaneous symmetry breaking

April 16

Today we will study a modern perspective on the nature of the hydro EFT, focusing on the simplest theory of diffusion (Lecture 5) and pure superfluidity (lecture 24).

$$\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_\phi \partial_t \phi \underbrace{- T(\pi_\rho \mu_\phi - \pi_\phi \mu_\rho)}_{\text{Poisson brackets (dissipationless)}} + \underbrace{iT\sigma \partial_i \pi_\rho \partial_i (\pi_\rho - i\mu_\rho)}_{\text{dissipative}} + \cdots$$

Observe that w/ only the terms in $\mathcal{L}_{MSR}$ above, π_ϕ is Lagrange multiplier

$$\frac{\partial \mathcal{L}_{MSR}}{\partial \pi_\phi} = 0 = \boxed{\partial_t \phi + T\mu_\rho}$$

Idea: invert this formula to find ρ in terms of $\partial_t \phi$

Now: $\mathcal{L} = -\rho(\partial_t\phi)\partial_t \pi_\rho - T\pi_\rho \mu_\phi + iT\sigma \partial_i \pi_\rho \partial_i(\pi_\rho + i\beta\partial_t\phi)$ $\quad (\beta = \frac{1}{T})$

Re-label: $\phi \to \phi_r$ and $\pi_\rho \to \phi_a$.

Since $T\mu_\phi = -\partial_i \tilde{J}$ due to $\phi \to \phi + c$ shift symmetry:

$$\mathcal{L} = -\partial_t \phi_a \cdot \rho(\partial_t \phi_r) - \partial_i \phi_a \cdot \tilde{J} + iT\sigma \partial_i \phi_a \partial_i(\phi_a + i\beta \partial_t \phi_r)$$

Now let's recall the rules from lecture 24:

Normal fluid: $\mu_\phi = 0 \Rightarrow \tilde{J} = 0$

Superfluid: $T\mu_\phi = -\partial_i(K\partial_i\phi) + \cdots \Rightarrow \tilde{J} = K\partial_i\phi_r + \cdots$

From general thermodynamics: $\rho = \rho_0 + \chi T\mu_\rho + \cdots \to \rho_0 - \chi\partial_t\phi_r + \cdots$

Leading order EFT is:

$$\mathcal{L} = \chi\,\partial_t\phi_a\partial_t\phi_r - K\partial_i\phi_a\partial_i\phi_r + iT\sigma\,\partial_i\phi_a\partial_i(\phi_a + i\beta\partial_t\phi_r)$$

Time-reversal symmetry? $\phi_a \to -\phi_a - i\beta\partial_t\phi_r$

$\phi_r \to -\phi_r$ ← so $\partial_t\phi_r$ invariant!

So far we've just rewritten the (SF) hydro EFT in new variables. But these new variables lead to $\mathcal{L}$ w/ interesting symmetries!

Symmetries: $\phi_a \to \phi_a + 1$

$\phi_r \to \phi_r + 1$ (superfluid) or $\phi_r \to \phi_r + f(x)$ (normal fluid)

Now it may seem too good to be true, but recall analogy from lec. 24:

SSB of a "ϕ_r U(1)" "ϕ_r U(1)" unbroken?

Taking a leap of faith we might indeed postulate that:

Hydro EFT has 2 U(1) symmetries "a" and "r":

"a" → always SSB

"r" → SSB in superfluid; unbroken in normal

But what could these a & r U(1)'s be? To answer this question we need to take a bit of a detour...

Hydro EFT is dissipative → <u>open</u> (quantum) system interacting w/ environment?

The quantum language will actually be <u>helpful</u> here.

In a pure quantum system, notion of symmetry:

pure state $|\psi\rangle$ and $\underbrace{\text{symmetry group } G}$

$\hookrightarrow$ unitary $U(g)$ for each $g \in G$.

$|\psi\rangle$ explicitly breaks G if $U(g)|\psi\rangle \neq e^{i\alpha}|\psi\rangle$ for any $g \in G$

constant phase

Instead we will diagnose SSB by presence of long-range order:

$$\lim_{|x-x'|\to\infty} \langle\psi| e^{i\phi(x)} \underbrace{e^{-i\phi(x')}} |\psi\rangle > 0$$

under $U(1)$ symmetry: $\phi(x) \to \phi(x) + c$

Let's look at a simpler example using discrete symmetry group:

$G = \mathbb{Z}_2$: generated by $X_1 \cdots X_n = U$

Pauli X: $|1\rangle\langle 0| + |0\rangle\langle 1|$.

Consider $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|000\cdots\rangle + |111\cdots\rangle\right]$. $U|\psi\rangle = |\psi\rangle$.

Consider $\langle\psi| Z_i Z_j |\psi\rangle = 1$ while $\langle\psi| Z_i |\psi\rangle = 0$

$\hookrightarrow$ Pauli $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

$\hookrightarrow$ Z charged under $\mathbb{Z}_2$.

$\Rightarrow |\psi\rangle$ has SSB.

Trace out one qubit: $\rho_{1\cdots N-1} = \mathrm{tr}_N |\psi\rangle\langle\psi|$

$$= \frac{1}{2}\underbrace{\left[|00\cdots\rangle\langle 00\cdots| + |11\cdots\rangle\langle 11\cdots|\right]}_{\text{a mixture of explicitly broken pure states}}$$

So we deduce that SSB is diagnosed by long-range quantum superpositions invisible to any few-body observables.

Now let's think about an analogy in statistical mechanics.

Consider two thermal ensembles:

$$\rho_c(\mu) = \sum_{Q} \sum_{\alpha} e^{-\mu Q} |\alpha\rangle\langle\alpha|$$ (canonical ensemble)

(don't worry about normalization)

total charge $Q = \sum Z_i$

all microstates w/ fixed Q

$$\rho_{mc}(Q_0) = \sum_{\alpha} |\alpha\rangle\langle\alpha|$$ (microcanonical ensemble)

α at fixed charge Q_0

Stat mech: w/ many qubits $N \gg 1$,

$$\mathrm{tr}_{(\frac{N}{2}+1)\dots N}\, \rho_{mc}(Q_0) \approx \mathrm{tr}_{1\dots\frac{N}{2}}\, \rho_c(\mu_0)$$

μ_0 and Q_0 are chosen so $\langle Z_i \rangle$ match between ensembles

This looks a lot like the SSB property we saw before... but what are the symmetries?

Density matrix ρ (mixed state) can have 2 kinds of symmetries:

Strong: $e^{i\alpha}\rho = U(g)\rho$ vs. Weak: $\rho = U(g)\,\rho\, U(g)^{-1}$

Note: strong symmetry $\Rightarrow$ weak symmetry

For $U(1)$ symmetry: $U(g) \rightarrow e^{i\theta Q}$ for constant θ.

Check: $$e^{i\theta Q}\rho_{mc}(Q_0) = e^{i\underbrace{\theta Q_0}_{\text{constant!}}}\rho_{mc}(Q_0) \Rightarrow \text{strong}$$

$$e^{i\theta Q}\rho_c = e^{i\theta Q}\cdot e^{-\mu Q} \neq e^{-\mu Q}$$

but $$e^{i\theta Q}e^{-\mu Q}e^{-i\theta Q} = e^{-\mu Q} \Rightarrow \text{weak}$$

Claim: microcanonical ensemble has SWSSB:
strong $\to$ weak spontaneous symmetry breaking

Now let's go back to understand the hydro EFT...
Hydro has charge-conserving dynamics, so EFT must have strong/weak U(1)s

Claim: ϕ_a is Goldstone of strong symmetry
$\Rightarrow$ $\phi_a \to \phi_a + 1$ only
ϕ_r is Goldstone of weak symmetry
$\Rightarrow$ $\phi_a \to \phi_a + f(x)$ for normal fluid
$\to \phi_a + 1$ for superfluid.

This explains the symmetry we found earlier. It's not trivial (and we won't cover in this class) but you could have built the hydro EFT using only knowledge of these SSB patterns!

If theory can exchange particles w/ bath:

$\Rightarrow$ strong U(1) explicitly broken

$$\mathcal{L} \to \chi\, \partial_t\phi_a \partial_t\phi_r - K \partial_i\phi_a \partial_i\phi_r + i T\sigma\, \partial_i\phi_a \partial_i(\phi_a + i\beta\partial_t\phi_r)$$
$$+\, i\Gamma T \phi_a(\phi_a + i\beta\partial_t\phi_r)$$

$\hookrightarrow$ EOM: $-\chi\partial_t^2\phi_r - \Gamma\partial_t\phi_r + \sigma\,\partial_i\partial_i\partial_t\phi_r = 0$

or: $\Gamma\mu + \chi\partial_t\mu = \sigma\nabla^2\mu$

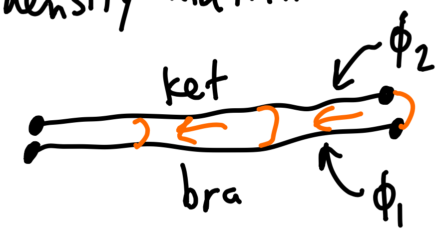
$$\Rightarrow \omega = -i\left(\boxed{\frac{\Gamma}{\chi}} + \frac{\sigma}{\chi}k^2\right)$$

gapped out hydro mode!

So the existence of diffusion as a long wavelength excitation is linked to the strong sym. being explicitly, not spontaneously, broken.

The EFT written in terms of ϕ_a & ϕ_r was understood before the SWSSB was discovered. A brief history...

Path integral for a density matrix:

$U \rho U^\dagger \longrightarrow$ 

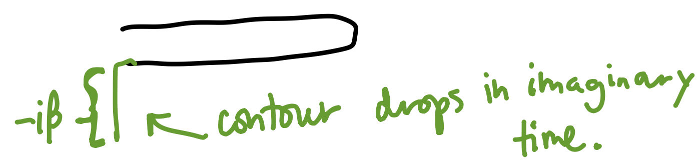
$\longrightarrow e^{iS[\phi_1, \phi_2]}$

Fields: $\phi_r = \dfrac{\phi_1 + \phi_2}{2}$ and $\phi_a = \dfrac{\phi_1 - \phi_2}{2}$.

unitarity: $S[\phi_r, \phi_a = 0] = 0$.

Path integral converges: $\mathrm{Im}(S) \geq 0$ (similar to lecture 4)

If density matrix is thermal:



Time-reversal symmetry comes from KMS symmetry:

$$\mathrm{tr}\left(e^{-\beta H} A(t) B(t')\right) = \mathrm{tr}\left(e^{-\beta H} B\left(t' - \tfrac{i\beta}{2}\right) A\left(t + \tfrac{i\beta}{2}\right)\right)$$

which is implemented via:

$$\phi_1(t) \to \pm\left[\phi_1\left(-t + \tfrac{i\beta}{2}\right)\right] \quad \text{and} \quad \phi_2(t) \to \pm\left[\phi_2\left(-t - \tfrac{i\beta}{2}\right)\right].$$

sign of field under T

In the hydro EFT:

$$\phi_r \to -\phi_r - \tfrac{i}{4}\beta \partial_t \phi_a$$

this correction not included by many authors

$$\boxed{\phi_a \to -\phi_a - i\beta \partial_t \phi_r}$$

matched onto time-reversal from the MSR Lagrangian