PHYS 7810
Hydrodynamics
Spring 2026

Lecture 27

Higher-form symmetry

April 21

Let's briefly recap the simple theory of diffusion (from MSR):

$\mathcal{L} = \pi \partial_t \rho + i\sigma \partial_i \pi \partial_i(\pi - i\mu)$ where $\mu \propto \rho$ in linear response

charge conservation from $\pi \to \pi + 1$ symmetry

But in some theories we have more interesting conserved quantities:

$Q_f = \int d^d x \, f(x) \rho(x)$ has $\frac{dQ_f}{dt} = 0$?

Noether's Theorem: $\pi \to \pi + f$ symmetry

Example 1: dipole conservation.

$Q = \int d^d x \, \rho(x)$ and $D_i = \int d^d x \, \rho(x) x_i$ conserved.

$\pi \to \pi + a + b_i x_i$ for constants a, b_i

Invariant building block is $\partial_i \partial_j \pi$

and $\mathcal{L} = \pi \partial_t \rho + iB \, \partial_i \partial_j \pi \, \partial_i \partial_j (\pi - i\mu)$ → $\partial_i \partial_i \pi \, \partial_j \partial_j (\pi - i\mu)$ equivalent after integrating by parts

$B \geq 0$ is hydrodynamic transport coefficient

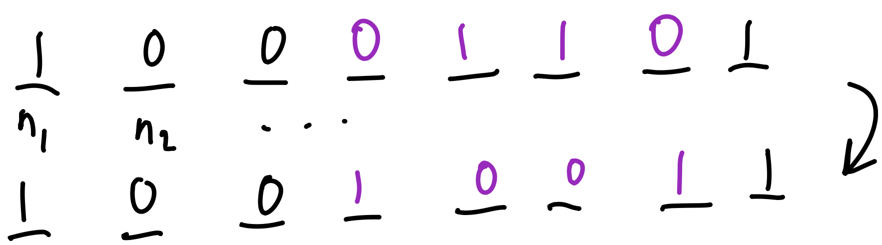
Noise free EOM: $\partial_t \rho + \partial_i \partial_j (B \partial_i \partial_j \mu) = 0$

or w/ $\mu = \frac{\rho}{\chi}$: $\partial_t \rho + D_4 \nabla^2 \nabla^2 \rho = 0$ where $D_4 = \frac{B}{\chi}$

Quasinormal modes: $\rho \sim e^{ikx - i\omega t} \Rightarrow$ $\boxed{\omega = -i D_4 k^4}$

Subdiffusion.

A microscopic setting where this theory arises is in "fracton hydrodynamics": consider particles moving on a lattice:

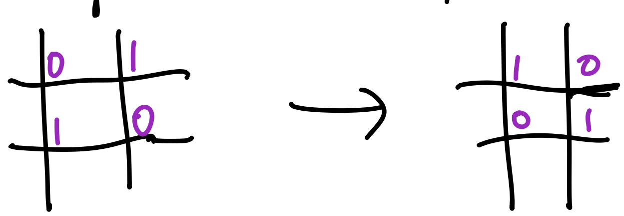


where we only allow microscopic moves conserving Q and D;

$Q = \sum n_x$ and $D = \sum x n_x$

Markov chain simulations indeed find subdiffusion!

Example 2: subsystem symmetries:



charge conserved in each row & column of 2d square lattice.

↳ continuum limit: $\int_{x=x_0} dy\, \rho(x,y)$ & $\int_{y=y_0} dx\, \rho(x,y)$ conserved.

or: $\int dx dy\, \rho(x,y) \cdot [f(x) + g(y)]$ for any f, g.

↳ $\pi \to \pi + f(x) + g(y)$.

Invariant motif: $\partial_x \partial_y \pi$

S_0: $\mathcal{L} = \pi \partial_t \rho + iB' \partial_x \partial_y \pi \, \partial_x \partial_y (\pi - i\mu)$

Following similar steps we find:

$$\omega = -iD_s k_x^2 k_y^2 \qquad \text{where } D_s = \frac{B'}{\chi}$$

not rotation invariant → lattice broke rotational symmetry

We can also consider theories where charge is not a scalar but itself transforms under spatial symmetry.

Suppose $\int d^d x \, \rho_i$ is conserved where i is spatial index:

$$\hookrightarrow \mathcal{L} = \pi_i \partial_t \rho_i + i\sigma_T \partial_i \pi_i \, \partial_j (\pi_j - i\mu_j)$$
$$+ i\sigma_S \frac{\partial_i \pi_j + \partial_j \pi_i}{2} \, \frac{\partial_i(\pi_j - i\mu_j) + \partial_j(\pi_i - i\mu_i)}{2}$$
$$+ i\sigma_A \frac{\partial_i \pi_j - \partial_j \pi_i}{2} \, \frac{\partial_i(\pi_j - i\mu_j) - \partial_j(\pi_i - i\mu_i)}{2}$$

broke up $\partial_i \pi_j$ into different irreps.

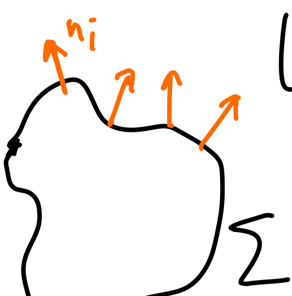
Focus on $d=3$. What if some of the $\sigma_T, \sigma_S, \sigma_A$ vanish?

If only $\sigma_A \neq 0$: motif $\partial_i \pi_j - \partial_j \pi_i$

invariant under $\pi_i \to \pi_i + \partial_i \Lambda$ for any $\Lambda(x)$:

So $Q_\Lambda = \int d^3 x \; \rho_i(x) \partial_i \Lambda(x)$ is conserved

Trial function: $\Lambda(x) = \begin{cases} 1 & x \in \Sigma \\ 0 & x \notin \Sigma \end{cases}$ for spatial region Σ.



$$\hookrightarrow Q_\Lambda = \oint_{\partial \Sigma} dA_i \, \rho_i = \text{flux of } \vec{\rho} \text{ through } \Sigma.$$

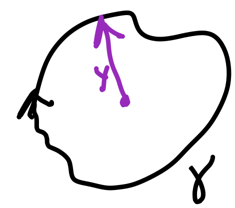
So all flux of ρ_i is conserved!

If only $\sigma_T \neq 0$: motif $\underbrace{\partial_i \pi_i}$

invariant under $\pi_i \to \pi_i + \varepsilon_{ijk}\partial_j \Lambda_k$

So: $\int d^3x \; \varepsilon_{ijk}\partial_j \Lambda_k \rho_i$ is constant.

Choose: $\Lambda_k = \oint_{\text{curve } \gamma} \varepsilon_{ijk} dy_j \, \partial_i \frac{1}{|x-y|}$



$\partial_i \oint d \frac{1}{|x-y|} = 0$

Biot - Savart: $\varepsilon_{ijk}\partial_j \Lambda_k = \oint_\gamma \left(dy_j \partial_j \partial_i \frac{1}{|x-y|} - dy_i \partial_j \partial_j \frac{1}{|x-y|} \right)$

$$\sim \oint_\gamma dy_i \, \delta(x-y)$$

So the conserved quantity is $\oint_\gamma dy_i \, \rho_i(y)$

There's an elegant way to understand/generalize these examples.
They are illustrations of a higher-form symmetry:

0-form symmetry (ordinary charge): $\underbrace{\partial_t \rho + \partial_i J_i}_{\partial_\mu J^\mu} = 0$

where $\mu = (t, x)$ & $J^\mu = (\rho, J_i)$.

p-form symmetry: $\partial_\mu J^{[\mu \nu_1 \cdots \nu_p]} = 0$

where $[\mu \nu_1 \cdots \nu_p]$ means indices fully antisymmetric.

d=3 and p=1: $\partial_t \underbrace{J^{ti}}_{\rho_i} + \partial_j \underbrace{\frac{J^{ji} - J^{ij}}{2}}_{\text{antisymmetric}} = 0$

Write:

$\mathcal{L} = \pi_i \partial_t \rho_i - \frac{\partial_i \pi_j - \partial_j \pi_i}{2} J_{ij}(\pi, \rho)$ with $J_{ij} = -J_{ji}$:

EOM: $\partial_t \rho_i + \partial_j \frac{J_{ji} - J_{ij}}{2} = 0$

So 1-form symmetry ⇒ charge conserved on (d−1)-dim surface!

To see this in more general dimensions, also consider:

$$\frac{d}{dt}\int d^dx \, J^{ti}\partial_i\Lambda(x) = -\int d^dx \, \partial_j J^{ji}\partial_i\Lambda = \int d^dx \, \Lambda \, \partial_i\partial_j J^{ji} \rightarrow 0 \text{ since } J^{ji} = -J^{ij}$$

$$= 0.$$

$d=3$ and $p=2$: $\partial_t J^{t[ij]} + \partial_k J^{[kij]} = 0$

$J^{t[ij]} = \varepsilon_{kij}\hat{\rho}_k$ $\quad$ $J^{[kij]} = \varepsilon_{kij}\hat{J}$

$$\varepsilon_{kij}\left[\partial_t \hat{\rho}_k + \partial_k \hat{J}\right] = 0.$$

Comes from: $\mathcal{L} = \pi_i \partial_t \rho_i - \partial_i \pi_i \, \hat{J}(\pi, \mu)$

↳ $\partial_t \rho_i + \partial_i \hat{J} = 0.$

2-form symmetry ⇒ charge conserved on (d−2)-dim shape
↳ curves in $d=3$.

p-form symmetry ⇒ conservation law on (d−p)-dim subspace ($0 \le p < d$)
— consistent w/ rotational/relativistic symmetry.

Quasinormal modes of 1-form theory:

$$\mathcal{L} = \pi_i \partial_t \rho_i + \frac{i\sigma}{2}(\partial_i \pi_j - \partial_j \pi_i)\left(\partial_i(\pi_j - i\mu_j) - \partial_j(\pi_i - i\mu_i)\right)$$

Assume $\Phi = \int d^dx \, \frac{1}{2\chi}\rho_i\rho_i + \cdots$ so $\mu_i = \frac{1}{\chi}\rho_i$.

EOM: $\partial_t \rho_i - \partial_j\left[\frac{\sigma}{\chi}(\partial_j \mu_i - \partial_i \mu_j)\right] = 0$ + noise.

Write $D = \frac{\sigma}{\chi}$: $\quad \partial_t \rho_i = D\left[\partial_j \partial_j \rho_i - \partial_i \partial_j \rho_j\right]$.

Suppose $\rho_i \sim e^{ikx - i\omega t}$:

If $i = x$: $\quad -i\omega \rho_x = -Dk^2(\rho_x - \rho_x) = 0$, so $\omega = 0$

If $i \perp x$: $\quad -i\omega \rho_\perp = -Dk^2 \rho_\perp$ since $\partial_i \rho_i = 0$.

So $\omega = -iDk^2$.

So 1-form charge can only diffuse perpendicular to the wave number: