PHYS 7810
Hydrodynamics
Spring 2026

Lecture 28
Magnetohydrodynamics

April 23

Recall: $d=3$, 1-form symmetry:

$$\mathcal{L} = \pi_i \partial_t \rho_i + \frac{i\sigma}{2} \underbrace{(\partial_i \pi_j - \partial_j \pi_i)}_{} \big(\partial_i(\pi_j - i\mu_j) - \partial_j(\pi_i - i\mu_i)\big)$$

$\pi_i \to \pi_i + \partial_i \Lambda(x)$ implies $\oint_\Sigma dA_i \rho_i$ conserved

In today's lecture we'll see an important application in physics:

Magnetism: $\partial_i B_i = 0$ and $\varepsilon_{ijk}\partial_j E_k + \partial_t B_i = 0$.

$$0 = \boxed{\partial_t B_i + \partial_j J_{ji}} \quad \text{where} \quad \boxed{J_{ji} = -J_{ij} = \varepsilon_{ijk} E_k}$$

magnetic field is 1-form charge

current is electric field

In a metal we might approximate that:

Ohm's Law: $E_k = \frac{1}{\sigma} J^{el}_k = \frac{1}{\sigma\mu_0}\varepsilon_{klm}\partial_l B_m$

electrical current

Ampere's Law

$$0 = \partial_t B_i + \frac{1}{\sigma\mu_0}\varepsilon_{ijk}\partial_j \varepsilon_{k\ell m}\partial_\ell B_m$$

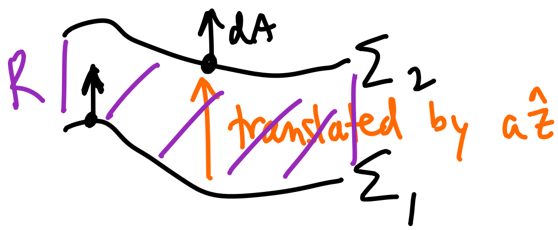
$$= \partial_t B_i + \frac{1}{\sigma\mu_0}\left[\partial_i\partial_j B_j - \partial_j\partial_j B_i\right]$$

This is the diffusion equation for 1-form charge that we found in lecture 27.

Magnetohydrodynamics (MHD) = 1-form charge + momentum

Applicable to motion of conducting fluids, astrophysics, etc.

Observe: $\partial_i B_i = 0$ crucial to consistency of theory:



$$Q_1 = \int_{\Sigma_1} dA_i\, B_i \quad \& \quad Q_2 = \int_{\Sigma_2} dA_i\, B_i$$

If $Q_1 \neq Q_2$ then $\boxed{\{P_z, Q_\Sigma\} \neq 0}$ in general

SSB if expected value of a Poisson bracket doesn't vanish.

But if $\partial_i B_i = 0$:

$$0 = \int_R d^3x\, \partial_i B_i = \oint_{\partial R} dA_i B_i = \int_{\Sigma_2} dA_i\, B_i - \int_{\Sigma_1} dA_i\, B_i, \quad \text{so} \quad Q_1 = Q_2.$$

Hence we don't need to worry about SSB.

Now let's think about using MSR framework to derive hydro:

$$\mathcal{L} = \pi_i \partial_t B_i + \sigma_i \partial_t g_i - \varepsilon_{ijk}\partial_j \pi_i E_k - \partial_j \sigma_i \tau_{ji} + \mathcal{L}_{diss}$$

Follow Lecture 19 to look for motifs that contribute to constitutive relations within ideal hydrodynamics.

Let $\mu_i = \frac{\delta \Phi}{\delta B_i}$ and $v_i = \frac{\delta \Phi}{\delta g_i}$.

Assume B_i, g_i both T-odd. Then we need:

$$\varepsilon_{ijk} \partial_j \mu_i \cdot E_k + \partial_j v_i \tau_{ji} = \partial_i S_i + R \partial_i B_i \ ?$$

since $\partial_i B_i = 0$ ...

We'll see that $R \neq 0$ necessary to make thermodynamics work. This derivation based on upcoming work w/ B. Bobell.

Note: $$\partial_i S_i + R \partial_i B_i = \partial_i \mu_j \underbrace{\left(\frac{\partial S_i}{\partial \mu_j} + R \frac{\partial B_i}{\partial \mu_j} \right)} + \partial_i v_j \left(\frac{\partial S_i}{\partial v_j} + R \frac{\partial B_i}{\partial v_j} \right)$$

$$\stackrel{?}{=} \partial_i \mu_j \underbrace{\left(- \varepsilon_{ijk} E_k \right)} + \partial_i v_j (\tau_{ij})$$

Need: $$\frac{\partial S_i}{\partial \mu_j} + \frac{\partial S_j}{\partial \mu_i} + R \left(\frac{\partial B_i}{\partial \mu_j} + \frac{\partial B_j}{\partial \mu_i} \right) = 0$$

$$\hookrightarrow = 2R \frac{\partial^2 \Phi}{\partial \mu_i \partial \mu_j} = \text{symmetric.}$$

Try: $S_i = -RB_i + A\mu_i + Cv_i$? Then:

$$0 = -B_i \frac{\partial R}{\partial \mu_j} - B_j \frac{\partial R}{\partial \mu_i} + 2\delta_{ij} A + \frac{\partial C}{\partial \mu_j} v_i + \frac{\partial C}{\partial \mu_i} v_j$$

Project into vector orthogonal to both B & v: $A = 0$.

Solution to other equations: $R = \mu_i v_i$ and $C = \Phi$ (pressure!)

$$0 = -B_i v_j - B_j v_i + B_j v_i + B_i v_j$$

Therefore: $-\varepsilon_{ijk} E_k = -B_i v_j + B_j v_i$ or $E_k = \varepsilon_{ijk} B_i v_j$

$$\tau_{ij} = \Phi \delta_{ij} + v_i v_j \frac{\partial \Phi}{\partial v^2/2} - B_i \mu_j$$

Interpret: $\vec{0} = \underbrace{\vec{E} + \vec{v}\times\vec{B}}_{\text{local electric field in rest frame of fluid vanishes}}$

$$\Phi = \int d^3x \left[\frac{g_i^2}{2\mathcal{M}} + \frac{B_i^2}{2\mu_0}\right]$$

$$\hookrightarrow \tau_{ij} = \Phi_{\text{fluid}}\,\delta_{ij} + \mathcal{M} v_i v_j + \left[\frac{B^2}{2\mu_0}\delta_{ij} - \frac{B_i B_j}{\mu_0}\right]$$

B-independent term

Maxwell stress tensor from E&M

In this limit it looks as if we can naively couple Navier-Stokes w/ Maxwell. This is the approximation used historically.

Quasinormal modes: $B_i = \bar{B}_i + \delta B_i e^{ikx - i\omega t}$ & $v_i = \delta v_i e^{ikx-i\omega t}$

const.

For simplicity we assume $v_i = 0$ around our static equilibrium:

$$\delta g_i = \mathcal{M}\delta v_i, \quad \delta\mu_i = \frac{\delta B_i}{\mu_0}, \quad \delta\Phi = \bar{B}_i \frac{\delta B_i}{\mu_0}$$

$$-i\omega\delta B_i + i\varepsilon_{ijk}k_j\varepsilon_{lmk}\bar{B}_l\delta v_m = 0$$

$$\text{or} \quad -i\omega\delta B_i + i\bar{B}_i k_j \delta v_j - i\,\delta v_i k_j \bar{B}_j = 0$$

$$-i\omega\mathcal{M}\delta v_i + ik_i\frac{\bar{B}_j\delta B_j}{\mu_0} - ik_j\left[\frac{\bar{B}_i\delta B_j + \bar{B}_j\delta B_i}{\mu_0}\right] = 0.$$

Since $\partial_i B_i = 0$, $\delta B_x = 0$.

Let $\bar{B}_i = \bar{B}(\cos\theta, \sin\theta, 0)$. Then:

$$\frac{\omega}{k}\begin{pmatrix}\delta B_y \\ \delta B_z \\ \delta v_x \\ \delta v_y \\ \delta v_z\end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{B}\sin\theta & -\bar{B}\cos\theta & 0 \\ 0 & 0 & 0 & 0 & -\bar{B}\cos\theta \\ c\sin\theta & 0 & 0 & 0 & 0 \\ -c\cos\theta & 0 & 0 & 0 & 0 \\ 0 & -c\cos\theta & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}\delta B_y \\ \delta B_z \\ \delta v_x \\ \delta v_y \\ \delta v_z\end{pmatrix}$$

where $c = \frac{\bar{B}}{\mu_0 M}$. Eigenvalues:

$\frac{\omega}{k} = 0$: $\delta\vec{v} \sim (\cos\theta, \sin\theta, 0)$ $[\delta\vec{v} \parallel \vec{\bar{B}}]$

$\frac{\omega}{k} = \sqrt{\bar{B}c} = v_A = \frac{\bar{B}}{\sqrt{\mu_0 M}}$: $\delta\vec{B}$, $\delta\vec{v} \perp \vec{\bar{B}}$

Alfvén wave:

$\frac{\omega}{k} = v_A \cos\theta$: $\delta B_z, \delta v_z \neq 0$: $\delta\vec{v}, \delta\vec{B} \perp \vec{k}$ & $\vec{\bar{B}}$.

Now let's look at dissipative corrections to MSR Lagrangian.

Assume: $B_i \to -B_i$ is symmetry of MSR.

$$\mathcal{L}_{diss} = i\eta_{ijkl}\partial_i\sigma_j\partial_k(\sigma_l - iv_l) + i\kappa_{ij}\varepsilon_{ikl}\varepsilon_{jmn}\partial_k\pi_l\partial_m(\pi_n - i\mu_n)$$

$$\kappa_{ij} = \kappa_1\delta_{ij} + \kappa_2\frac{\bar{B}_i\bar{B}_j}{\bar{B}^2}$$ if expanding around same solution as before.

interpret as electrical resistivity as at beginning of class.

Assume rotational symmetry, and T: $\eta_{ijkl} = \eta_{jikl} = \eta_{klij}$

$$\eta_{ijkl} = \eta\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right] + \zeta\delta_{ij}\delta_{kl}$$
$$+ \eta_{\parallel}\frac{\bar{B}_i\bar{B}_j\bar{B}_k\bar{B}_l}{\bar{B}^4} + \tilde{\eta}_{\parallel}\left(\delta_{ij}\frac{\bar{B}_k\bar{B}_l}{\bar{B}^2} + \delta_{kl}\frac{\bar{B}_i\bar{B}_j}{\bar{B}^2}\right)$$
$$+ \eta_m\left[\delta_{ik}\frac{\bar{B}_j\bar{B}_l}{\bar{B}^2} + \delta_{il}\frac{\bar{B}_j\bar{B}_k}{\bar{B}^2} + \delta_{jk}\frac{\bar{B}_i\bar{B}_l}{\bar{B}^2} + \delta_{jl}\frac{\bar{B}_i\bar{B}_k}{\bar{B}^2}\right]$$

Usually the EOMs of MHD are derived by a basic coupling of Maxwell to Navier-Stokes. This symmetry-based perspective shows transport coefficients missing in standard analysis...