PHYS 7810
Hydrodynamics
Spring 2026

Lecture 3
Time-reversal symmetry

January 15

Stochastic differential equation: $\dot{x}_i = a_i(x) + b_{i\alpha}(x)\xi_\alpha(t)$

with $\langle \xi_\alpha(t)\rangle = 0$, $\langle \xi_\alpha(t)\xi_\beta(s)\rangle = \delta_{\alpha\beta}\delta(t-s)$

↳ Fokker-Planck equation: $\partial_t P(\vec{x},t) = -\partial_i(a_i P) + \frac{1}{2}\partial_i\partial_j(b_{i\alpha}b_{j\alpha}P)$

In lecture 2 we saw that sometimes $P(x,t)$ relaxes to a steady-state distribution at late times:

$$\lim_{t\to\infty} P(\vec{x},t) = P_{ss}(\vec{x}) \sim e^{-\Phi(\vec{x})} \qquad \text{[neglect normalization].}$$

Let's now think about things from reverse perspective. If we knew the form of $e^{-\Phi}$, what could we say about the FPE?

In fact it will be instructive to re-derive FPE from a more phenomenological, "effective theory" perspective:

If $\partial_t P = -\hat{W}P$, what can $\hat{w}$ be?

($\hat{W}$ ← formal differential operator)

① $\int dx_1 \cdots dx_n \, P = 1$ at all times: $\int dx_1 \cdots dx_n \, \partial_t P = 0$?

Since $P(\vec{x},t) \to 0$ as $|\vec{x}| \to \infty$, would hold if

$$\hat{W} P = \partial_i (\hat{A}_i P) \qquad \text{since} \qquad \int d^n x \left(-\partial_i(\hat{A}_i P)\right) = 0$$

$\hat{A}_i$ ← new differential op. — $= 0$ by divergence thm.

② stationarity: $\hat{W}(e^{-\Phi}) = 0.$ holds if

$$\hat{W} P = \hat{B}_j (\partial_j P + P \partial_j \Phi) = \hat{B}_j (\partial_j + \mu_j) P$$

↳ define $\mu_j = \partial_j \Phi$

since $(\partial_j + \mu_j) e^{-\Phi} = -\mu_j e^{-\Phi} + \mu_j e^{-\Phi} = 0.$

Now let's combine these arguments:

$$\hat{W} P = -\partial_i \hat{Q}_{ij} (\partial_j + \mu_j) P$$

$\hat{Q}_{ij}$ ← general differential operator

One can show that if x lives on a simply connected space, ALL $\hat{W}$ must take this form!

"effective theory" ⇒ expand $\hat{Q}$ in derivatives ∂_k ?

leading order: $\hat{Q}_{ij} = Q_{ij}(x)$ ⟹ Gaussian white noise!

In this class we will focus on this Gaussian noise limit...

This new perspective can bear further fruit if we make an analogy to QM. First, a mathematical analogy:

Green's function $G(x,t;y)$ solves FPE w/ $G(x,0;y) = \delta(x-y)$.

$G(x,t;y)$ ↓ probability density $y \to x$ in time t

$\delta(x-y) = \langle x | y \rangle$

So $G(x,t;y) = \langle x| e^{-\hat{W}t} |y\rangle$. Indeed if $P(x,t) = \langle x | P(t)\rangle$, FPE is $\frac{d}{dt}|P(t)\rangle = -\hat{W}|P(t)\rangle$.

In QM, we can do more:

$$\langle x| e^{-iHt} |y\rangle = \langle y| e^{-iHt} |x\rangle^*$$

i.e. amplitude of $y \to x$ matches $x \to y$ (up to phase)

because $H = H^\dagger$.

For FPE: $\langle x| e^{-\hat{W}t} |y\rangle = \langle y| e^{-\hat{W}^T t} |x\rangle$ (linear algebra on real vector space...)

but $\hat{W}^T$ is NOT generator of another FPE!

How can we preserve structure of $\hat{W}$ under "transpose"?

Recall: $\hat{W} = \partial_i Q_{ij}(\partial_j + \mu_j) = \partial_i Q_{ij} e^{-\Phi} \partial_j e^{\Phi}$

since $e^{-\Phi}\partial_j(e^{\Phi} f) = e^{-\Phi} \cdot e^{\Phi}(\partial_j f + \mu_j f) = (\partial_j + \mu_j) f$

and since $(\partial_i)^T = -\partial_i$: (if Q doesn't have any ∂...)

$$\hat{W}^T = e^{\Phi}(-\partial_j) e^{-\Phi} Q_{ij}(-\partial_i) = e^{\Phi} \partial_j Q_{ji} e^{-\Phi} \partial_i$$

so $e^{-\Phi} \hat{W}^T e^{\Phi} = \partial_i Q_{ji} e^{-\Phi} \partial_j e^{\Phi} = \partial_i Q_{ji}(\partial_j + \mu_j) = \hat{W}_{rev}$ (time-reversed $\hat{W}$)

takes FPE form but Q has been transposed!

Conclusion: $\underbrace{\langle x| e^{-\hat{W}t} |y\rangle}_{\text{probability to transition from } y \to x \text{ in time } t} = \langle y| e^{-\hat{W}^T t} |x\rangle$

$$= \langle y| e^{\Phi} e^{-\hat{W}_{rev} t} e^{-\Phi} |x\rangle$$

$$= e^{\Phi(y) - \Phi(x)} \underbrace{\langle y| e^{-\hat{W}_{rev} t} |x\rangle}_{\text{prob. } x \to y \text{ under reversed process}}$$

Time-reversal symmetry: $\hat{W} = \hat{W}_{rev}$.

$\Downarrow$

Detailed balance: $\langle x | e^{-\hat{W}t} | y \rangle e^{-\Phi(y)} = \langle y | e^{-\hat{W}t} | x \rangle e^{-\Phi(x)}$

Now let's see that time-reversal was behind FDT:

$\hat{W} = -\partial_i Q_{ij} (\partial_j + \mu_j)$, T-symmetric: $Q_{ij} = Q_{ji}$

$\hookrightarrow \partial_t P = -\partial_i \left[\underbrace{(Q_{ij}\mu_j + \partial_j Q_{ij})}P \right] + \partial_i \partial_j \left[Q_{ij} P \right]$

fluctuation-dissipation: $Q_{ij}\mu_j$ dissipative force related to noise

Let's see how this reconciles our confusion about time-reversal symmetry vs. arrow of time in dissipative systems:

Example 1: $\dot{x} = -\mu x + \sigma \xi(t)$

lec 2 $\hookrightarrow e^{-\Phi} = e^{-\mu/\sigma^2 x^2}$ $\rightarrow$ Define $\xi = \sigma/\sqrt{\mu}$.

DB: $\mathbb{P}(x_0 \rightarrow 0) \boxed{e^{-(x_0/\xi)^2}} = \mathbb{P}(0 \rightarrow x_0)$

arrow of time! very unlikely to go to atypical configuration

Sometimes it's helpful to include other transformations as part of TRS:

Example 2: Brownian motion.

$\dot{x} = p/m$ & $\dot{p} = \alpha \xi(t)$ (see also HW 1)

$\alpha = 0$: $T \cdot x = x$ & $T \cdot p = -p$. (momenta are odd under T)

Fokker–Planck $\rightarrow$ Liouville equation in dissipationless limit:

$$\partial_t P + \partial_x\left(\frac{p}{m}P\right) + \cancel{\partial_p(0\cdot P)} = 0, \quad \text{i.e.} \quad \hat{W} = \partial_x\left(\frac{p}{m}\cdot\right)$$

under T: $\hat{W}_{rev} = (-p/m)(-\partial_x) = \partial_x(p/m\ \cdot) = \hat{W}$

Note: steady state has translation symmetry so no μ_x needed.

More generally could take "generalized" time-reversal:

$$\hat{W}_{rev} = e^{-\Phi}\,\mathcal{T}\,\hat{W}^T\,\mathcal{T}e^{\Phi} \qquad \text{where } \mathcal{T}^2 = 1,\ \mathcal{T}\cdot\Phi = \Phi$$

In our example:

$$\mathcal{T}\begin{pmatrix} x \\ p \\ \partial_x \\ \partial_p \end{pmatrix}\mathcal{T} = \begin{pmatrix} x \\ -p \\ \partial_x \\ -\partial_p \end{pmatrix}$$

"Generalized time-reversal symmetry": $\hat{W}_{rev} = \hat{W}$.

Often in physics we care about conserved quantities. Noether's Thm: symmetry $\Leftrightarrow$ conservation law in Hamiltonian mech. What to make of this in stochastic systems? A subtle thing is that there are actually 2 kinds of "symmetries..."

strong symmetry: $F(x_1,\ldots,x_n)$ conserved on every trajectory
weak: on average

Constraints on $\hat{W}$?

Strong: $[\hat{W}, F(x_1,\ldots,x_n)] = 0$ (analogy to QM!)

$$\Downarrow$$

$$e^{-\lambda F}\,\hat{W}\,e^{\lambda F} = \hat{W} \quad \text{for any } \lambda$$

$$\Downarrow$$

$$\hat{W}(x_i, \partial_i) = \hat{W}(x_i, \partial_i + \partial_i F)$$

Example 3: total momentum conservation

$$\dot{p}_1 = -\alpha(p_1 - p_2) + \beta\xi(t) \qquad \dot{p}_2 = -\alpha(p_2 - p_1) - \beta\xi(t)$$

$\hookrightarrow$ $\dot{p}_1 + \dot{p}_2 = 0$ Define $\bar{p} = p_1 + p_2$

Fokker-Planck equation:

$$\partial_t P = -\partial_{p_1}\left[-\alpha(p_1 - p_2)P\right] - \partial_{p_2}\left[-\alpha(p_2 - p_1)P\right] + \frac{1}{2}\left(\beta\partial_{p_1} \;\; -\beta\partial_{p_2}\right)\begin{pmatrix}\beta\partial_{p_1} \\ -\beta\partial_{p_2}\end{pmatrix}P$$

$$= -(\partial_{p_1} - \partial_{p_2})\left[\alpha(p_1 - p_2)P\right] + \frac{\beta^2}{2}(\partial_{p_1} - \partial_{p_2})^2 P$$

Shift: $\partial_{p_1} \to \partial_{p_1} + \frac{\partial \bar{p}}{\partial p_1} = \partial_{p_1} + 1$, $\quad \partial_{p_2} \to \partial_{p_2} + \frac{\partial \bar{p}}{\partial p_2} = \partial_{p_2} + 1$

$\partial_{p_1} - \partial_{p_2} \to \partial_{p_1} - \partial_{p_2}$ is invariant!

weak: $\frac{d}{dt}\int d^n x \; F(\vec{x})\, P(\vec{x}, t) = 0$

$\Downarrow$

$$\langle F | \frac{d}{dt} | P\rangle = -\langle F | \hat{W} | P\rangle = 0$$

$\Downarrow$

$$\langle F | \hat{W} = 0.$$

Modify:
$$\dot{p}_1 = -\alpha(p_1 - p_2) + \beta\xi_I + \gamma\xi_{II}$$
$$\dot{p}_2 = -\alpha(p_2 - p_1) - \beta\xi_I + \gamma\xi_{II}$$

$\hookrightarrow$ $\frac{d}{dt}\bar{p} = 2\gamma\xi_{II}$

So $\frac{d}{dt}\langle\bar{p}\rangle = 0$ but $\bar{p}$ fluctuates on typical trajectories