PHYS 7810
Hydrodynamics
Spring 2026

Lecture 4
MSR Lagrangian

January 20

So far we have seen how to build FPEs & enforce T.
This works but it's a bit cumbersome. Today's goal is to give a simplified Lagrangian (approximate) formulation of the same formalism.

Recap: Fokker-Planck equation w/ steady-state $e^{-\Phi}$:

$$\partial_t P(\vec{x},t) = -\hat{W}P = \partial_i\left[Q_{ij}(\partial_j + \mu_j)P\right] \quad \text{where} \quad \mu_j = \partial_j \Phi.$$

If $x_i(t) \to x_i(-t)$ under T, then theory is T-symmetric if $Q = Q^T$.

Goal: convert FPE $\to$ MSR Lagrangian
↳ Martin, Siggia, Rose (1973)

Inspiration for this will come from QM, where we can use path integrals to connect Schrödinger's eqn $\to$ Lagrangians.

QM transition amplitude: $\langle x | e^{-iHt} | y \rangle$ $\rightsquigarrow$ stochastic transition probability $\langle x | e^{-\hat{W}t} | y \rangle$

This will be the only place in the course where we use path integrals so don't worry if these manipulations aren't so familiar:

Idea: $\langle z| e^{-\hat{W}T} |y\rangle =$ probability (density) to transition from $y \to x$ after time t

$$= \Big\langle \underbrace{\int D x_i}_{\text{integrate over all paths w/ } x_i(0)=y_i \ \& \ x_i(T)=z_i} \delta[\dot{x}_i - a_i - b_{i\alpha}\xi_\alpha] \Big\rangle_{\xi_\alpha}$$

Heuristic: $D x_i = \prod_{A=1}^{T/\delta t} d x_i(A\delta t)$,

$$\delta[\dot{x} - \cdots] = \prod_A \delta\big(x_i((A+1)\delta t) - x_i(A\delta t) - \delta t\,(a_i + \cdots)\big)$$

Important! Use Itô prescription so that

$$1 = \int dz \, \langle z| e^{-\hat{W}T} |y\rangle = \int dz \int_{\substack{x_i(0)=y \\ x_i(T)=z}} D x_i \big\langle \delta[\dot{x}_i - \cdots] \big\rangle_\xi$$

because $\int d z_i \, \delta\big(z_i - x_i(T-\delta t) - \delta t \cdot (a_i(x_i(T-\delta t)) + \cdots)\big) = 1$

Crucially the integrand only has simple dependence on z_i.

Now we will need to use 2 more path integrals:

1) $\delta[F_i] = \int D\pi_i \, e^{i \pi_i F_i}$ $\quad i = \sqrt{-1}$ (straightforward generalization of Fourier transform of Dirac δ)

2) $\langle F \rangle_{\xi_\alpha} = \int D\xi_\alpha \cdot F \exp\left[-\int_0^T dt \, \frac{1}{2}\xi_\alpha \xi_\alpha\right]$ normalized for Gaussian white noise

Combine these ingredients together:

$$\langle z | e^{-\hat{W}T} | y \rangle = \int Dx_i D\pi_i D\xi_\alpha \exp\left[iS \right] \quad \text{where}$$

$$S = \int_0^T dt \left[\pi_i \dot{x}_i - \pi_i a_i(x) - \pi_i b_{i\alpha}(x) \xi_\alpha + \frac{i}{2} \xi_\alpha \xi_\alpha \right]$$

Now we can do the path integral over ξ_α explicitly to get:

$$\langle z | e^{-\hat{W}T} | y \rangle = \int Dx_i D\pi_i \, e^{i\int_0^T dt \, L_{MSR}},$$

$$L_{MSR} = \pi_i \dot{x}_i - \pi_i a_i(x) + \frac{i}{2} \pi_i \pi_j b_{i\alpha} b_{j\alpha}$$

If we've written $b_{i\alpha} b_{j\alpha} = \underbrace{Q_{ij} + Q_{ji}}_{\text{symmetric part} \Rightarrow \text{fluctuations}}$ and $a_i = -Q_{ij}\mu_j$ then:

$$L_{MSR} = \pi_i \dot{x}_i + i \pi_i Q_{ij} [\pi_j - i\mu_j]$$

Analogy to QM: in FPE, $\pi_i \to -i\partial_i$ (~ momentum)

Crucially however, L_{MSR} is complex-valued b/c FPE describes dissipative processes. Different to unitary QM...

Historically people often used tools from QFT to analyze stochastic equations via MSR. But we find it more helpful to instead build effective theories starting w/ MSR!

Rules for MSR Lagrangian effective theory...:

1) Identify (slow) DOF $\to x_i$. Introduce "partner" π_i.

2) Write $L_{MSR} = \pi_i \dot{x}_i - \mathcal{H}_{MSR}(\pi, x)$ and enforce:

$\mathcal{H}_{MSR}(\pi = 0, x) = 0$ (process is stochastic)

$\mathrm{Im}(\mathcal{H}_{MSR}) \leq 0$ (noise variance positive)

3) Strong symmetry: if F is conserved <u>before</u> noise average:

$\mathcal{H}_{MSR}$ invariant under $\pi_i \to \pi_i + \frac{\partial F}{\partial x_i}$

This is analogous to rule we wrote down in lecture 3!

4) Time-reversal symmetry: if steady-state Φ, $\mu_i = \partial_i \Phi$:

if $x_i \to \pm x_i$ then $\pi_i \to \mp(\pi_i - i\mu_i)$ (?)

↳ if all $x_i \to x_i$ then $Q_{ij} = Q_{ji}$ ✓

↳ however, we miss a subtle correction:

$$a_i = -Q_{ij}\mu_j + \underline{\partial_j Q_{ij}}$$

Similar issues arise in QM and come from challenge of deducing proper operator ordering in a path integral. For most practical purposes we ignore this and move on.

Another sanity check is that T reproduces the detailed balance condition at the level of the path integral!

$$\langle z | e^{-\hat{W}T} | y \rangle = \int_{\substack{x(0)=y \\ x(T)=z}} Dx\, D\pi\, \underbrace{e^{i\int_0^T dt\,(\pi_i \dot{x}_i - \mathcal{H})}}_{e^{iS}}$$

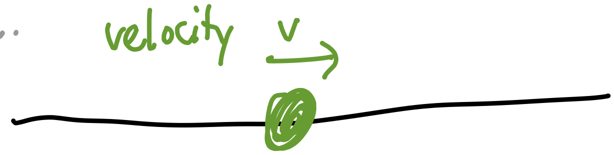
$$\hookrightarrow \int_{\substack{x(0)=y \\ x(-T)=z}} Dx\, D\pi\, e^{i\int_{-T}^{0} dt\,(\pi_i \dot{x}_i - \mathcal{H} - i\mu_i \dot{x}_i)} = \int_{\substack{x(0)=y \\ x(-T)=z}} Dx\, D\pi\, e^{iS + \Phi(y) - \Phi(z)}$$

$$= \langle y | e^{-\hat{W}T} | z \rangle e^{\Phi(y) - \Phi(z)} \quad ✓$$

which reproduces the detailed balance condition from before!

Now let's return to our puzzle from lecture 1 about the particle on a line.

Example 1:



Assume thermal equilibrium: $\Phi = \beta \cdot \frac{1}{2} m v^2$ ← $\beta m > 0$ so steady-state is normalizable

under T: $v \to -v$:

$$L_{MSR} = \pi \dot{v} + i\gamma \pi(\pi - i\mu) \qquad \text{where} \quad \mu = \beta m v$$

constant $\gamma > 0$ since noise has positive variance

Motif added to L_{MSR} above is the only one invariant under T:

Under T: $\pi(\pi - i\mu) \to (\pi - i\mu)\big((\pi - i\mu) - i(-\mu)\big) = (\pi - i\mu)\pi$

$\mu \to -\mu$ since v odd

EOM for v, neglecting noise:

$$\left.\frac{\delta S_{MSR}}{\delta \pi}\right|_{\pi=0} = 0 = \left.\left[\frac{\partial L}{\partial \pi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\pi}}\right]\right|_{\pi=0} = \dot{v} + \gamma\mu = \dot{v} + \underbrace{\gamma \beta m}\, v$$

positivity arose from basic principles!

So MSR framework solves all of our puzzles raised in lecture 1. In particular we now understand origin of positivity constraints in dissipative effective theories.

Fluctuation - dissipation theorem solves "arrow of time": $i\gamma\pi(\pi - i\mu)$, is manifest in MSR.

Example 2: damped spin dynamics.

Consider spin S_i ← $i = 1,2,3$ w/ Poisson brackets:

$$V_{ij} = \{S_i, S_j\} = \varepsilon_{ijk} S_k$$
$$= -V_{ji}$$

Levi-Civita

Hamilton's equations $\dot{S}_i = \{S_i, H\}$ become

$$L_{MSR} = \pi_i \dot{S}_i - \pi_i V_{ij} \mu_j$$

↳ since $V = -V^T$, no noise term!

This is a very general way to take a Hamiltonian mechanical system and "lift it" into the MSR framework!

Note: $\{S^2, \cdot\} = 0$: only orientation of spin can change.

Add dissipation that preserves length of spin?

Need $\mathcal{H}$ invariant under $\pi_i \to \pi_i + \partial_i(S^2) = \pi_i + 2S_i$.

invariant "building block": $\varepsilon_{ijk} S_j \pi_k \to \varepsilon_{ijk} S_j (\pi_k + \underbrace{\cancel{2S_k}}_{0 \text{ by Levi-Civita}})$

↳ Note: $\pi_i V_{ij} \mu_j = \pi_i \varepsilon_{ijk} S_k \mu_j$ made out of invariant!

Hence,

$$L_{MSR} = \pi_i \dot{S}_i - \pi_i \varepsilon_{ijk} S_k \mu_j + i\gamma \underbrace{[\varepsilon_{ijk} S_j \pi_k][\varepsilon_{ilm} S_l (\pi_m - i\mu_m)]}_{\text{invariant under } \pi \to \pi + 2S \text{ \& } T}$$

Noise free EOMs:

$$\dot{S}_i = \varepsilon_{ijk} \mu_j S_k - \underbrace{\gamma \varepsilon_{nji} \varepsilon_{nlm} S_j S_l \mu_m}_{= \gamma [S_j S_j \mu_i - S_i S_j \mu_j] = \gamma [S^2 \vec{\mu} - (\vec{S} \cdot \vec{\mu}) \vec{S}]}$$

The specific form of this motif might have been very hard to guess, but using MSR framework we have a systematic way to figure out the answer!