PHYS 7810
Hydrodynamics
Spring 2026

Lecture 5
Diffusion

January 22

lectures 2-4 developed Lagrangians for noisy/dissipative systems

stochastic eq for DOF $x_i \longrightarrow$ MSR Lagrangian:

$$L = \pi_i \dot{x}_i - \mathcal{H}(\pi, x) \quad \text{where} \quad \mathcal{H}(0,x) = 0, \quad \mathrm{Im}(\mathcal{H}) \leq 0$$

b/c process is stochastic

and $\mathcal{H}$ invariant under symmetries:

- time-reversal: $x_i \to \pm x_i$ & $\pi_i \to \mp(\pi_i - i\mu_i)$ $\quad \mu_i = \frac{\partial \Phi}{\partial x_i}$
- conserved quantity F (strong symmetry):
  $$\pi_i \to \pi_i + \frac{\partial F}{\partial x_i}.$$

And in lecture 1 we said that we needed this technology to systematically study hydrodynamics!

Hydrodynamics = dissipative effective field theory of systems approaching equilibrium

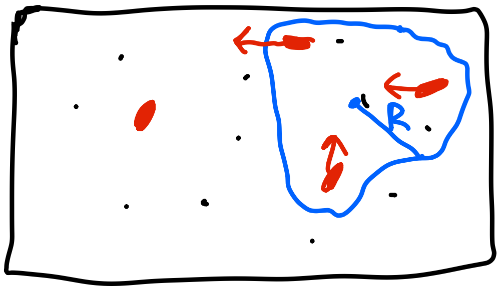
MSR Lagrangians

fields, not particles!

specific, slow DOF...

This lecture will build the simplest hydro EFT: the theory of diffusion of a single conserved charge...

Example:



dye molecules moving in water...

$$N = \int d^d x \; \rho(x,t) \quad \text{is conserved}$$

# of dye mols

density of dye mols

$\rho(\vec{x},t)$ should have slow dynamics (draw blue/red arrows in fig now)

→ only $\sim R^{d-1}$ molecules near boundary can leave

→ $\frac{\partial \rho}{\partial t} \lesssim \frac{R^{d-1}}{R^d} \sim \frac{1}{R} \to 0$ on large scales $R \to \infty$

Hydrodynamic postulate: slow DOF from conserved quantities

We'll provide more arguments for this in lecture 6 but it's postulate.

Related to intuition that stat mech can ignore most microscopic DOF.

Goal: MSR field theory for $\rho(x,t)$ directly?

Intuition: discretize $\rho(x,t) \to \begin{cases} \rho_1(t) = \rho(x_1,t) \\ \rho_2(t) = \rho(x_2,t) \\ \vdots \end{cases}$

$$\downarrow$$

$$L = \int d^d x \left[\pi(x) \partial_t \rho(x) - \mathcal{H} \right] \; ? \xleftarrow{\text{continuum}} L = \sum_{\text{points } a} \pi_a \dot{\rho}_a - \mathcal{H}(\pi,\rho)$$

$\downarrow$ It's conventional to write Lagrangian density

$$\mathcal{L} = \pi \partial_t \rho - \mathcal{H}(\pi,\rho) \quad \text{with action} \quad S = \int dt \, d^d x \, \mathcal{L}$$

Euler-Lagrange equations:

$$0 = \frac{\delta S}{\delta \rho} = \frac{\partial \mathcal{L}}{\partial \rho} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial_t \rho)} - \frac{\partial}{\partial x_i}\frac{\partial \mathcal{L}}{\partial(\partial_i \rho)} \quad \text{etc.}$$

← functional derivative

Rules for dealing with symmetries generalize straightforwardly to field theory.

Hydro postulate: $N = \int d^d x\, \rho$ is conserved

$$\Rightarrow \text{symmetry under } \pi(x) \to \pi(x) + \underbrace{\frac{\delta N}{\delta \rho}} = \pi(x) + 1$$

$$= \frac{\partial}{\partial \rho}(\rho) - \frac{\partial}{\partial t}\frac{\partial}{\partial(\partial_t \rho)}\rho - \cdots = 1$$

Time-reversal symmetry: $\rho \to \rho$ and $\pi \to -\pi + i\mu$.

But what should we take for the steady-state Φ? and μ?

Although we'll justify this choice more tomorrow, a natural guess is:

Locality of physics $\Rightarrow$ $\Phi = \int d^d x\, \phi(\rho)$

(linear term not needed: N is conserved!)

$$\hookrightarrow \phi(\rho) = \frac{\rho^2}{2\chi} + \cancel{\frac{\alpha}{3}\rho^3 + \cdots}$$

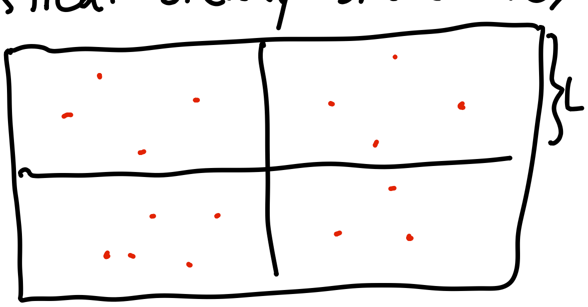
just keep leading

$\chi > 0$ so Φ normalizable:

$$\mu = \frac{\delta \Phi}{\delta \rho} \approx \frac{\rho}{\chi} \leftarrow \text{susceptibility.}$$

Here is a more microscopic argument for this choice of Φ:

Statistical steady-state $\Leftrightarrow$ thermal equilibrium ...:



each region has $\rho \cdot L^d$ dye mols...

central limit theorem:

Gaussian fluctuations $\sim \sqrt{\rho L^d}$ in each box

Gaussian fluct. $\Longrightarrow$ $\Phi \sim \int d^d x \, \rho^2$.

Finally we understand each input into MSR EFT. Let's build our hydrodynamic theory!

$\mathcal{L} = \pi \partial_t \rho - \mathcal{H}$ where $\mathcal{H}$ invariant under

$\mu \approx \rho/\chi$

$\pi \to -\pi + i\mu$

$\pi \to \pi + 1$

number cons.

Look for invariant building blocks under these motifs:

$\underbrace{\partial_x \pi}$

simplest, by locality in space

$\partial_x \pi \, \partial_x (\pi - i\mu)$

$\downarrow$

$\partial_x(-\pi + i\mu) \, \partial_x(-\pi + i\mu - i\mu)$

$= (-\partial_x \pi) \cdot (-\partial_x (\pi - i\mu))$

Since $\mathrm{Im}(\mathcal{H}) \leq 0$ we deduce:

$$\mathcal{L} = \pi \partial_t \rho + i\sigma(\rho) \, \partial_x \pi \, \partial_x(\pi - i\mu) + \cdots$$

$\sigma(\rho) \geq 0$ is a phenomenological parameter of hydro

$\to$ transport coefficient

As usual in effective theory we could add higher-order terms in ... and we are focusing on the leading order terms.

Vary with π to deduce EOMs:

0, neglecting noise for now...

$$\frac{\delta S}{\delta \pi} = 0 = \partial_t \rho - i\partial_x \left[\sigma(\rho) \, \partial_x (\cancel{2\pi} - i\mu) \right]$$

$$\partial_t \rho = \partial_x \left[\sigma(\rho) \, \partial_x \mu \right] \approx \sigma_0 \, \partial_x^2 \left(\frac{\delta \rho}{\chi} \right)$$

if $\rho \approx \rho_0 + \delta\rho(x)$

$\sigma_0 = \sigma(\rho_0)$

$$\partial_t \delta\rho = D \, \partial_x^2 \delta\rho$$

diffusion constant

Einstein relation

$$\underbrace{D = \frac{\sigma_0}{\chi}}$$

We already saw the solution to this equation in Lecture 2, but here the interpretation is different. ρ is the density of a conserved quantity, <u>not</u> the probability density of one particle undergoing a random walk!

$$\delta\rho(x,t) = \int_{-\infty}^{\infty} dy \, \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi D t}} \, \delta\rho(y,0)$$

↳ lump of charge spreads out over $\Delta x \sim \sqrt{Dt}$.

Alternately: quasinormal modes (plane wave solutions):

$$\delta\rho \sim e^{i[\vec{k}\cdot\vec{x} - \omega t]} \quad \longrightarrow \quad \underbrace{-i\omega \cdot \delta\rho = -D|\vec{k}|^2 \cdot \delta\rho}_{\omega = -iD|\vec{k}|^2}$$

$\mathrm{Im}(\omega) \leq 0 \rightarrow$ mode decays w/ time: dissipative...

It's helpful to go back and notice some general structure in the hydro EOMs...

In general: $\mathcal{L} = \pi \partial_t \rho - \partial_x \pi \cdot \tilde{J}(\pi, \rho)$

N conserved

π-dependence $\Rightarrow$ (stochastic) fluctuating hydro

Leading order EFT: $\tilde{J} = -i\sigma \partial_x (\pi - i\mu)$

Go from MSR $\longrightarrow$ Fokker-Planck $\longrightarrow$ Langevin:

$$\partial_t \rho + \partial_x J = 0$$

stochastic correction:

$$\langle \xi_J \rangle = 0$$

$$J = \underbrace{-\sigma \partial_x \frac{\rho}{\chi}}_{\substack{\text{constitutive relation} \\ \text{for } J \text{ in terms of } \rho}} + \xi_J(x,t) \qquad \langle \xi_J(x,t)\, \xi_J(x',t') \rangle = 2\sigma\, \delta(x-x')\,\delta(t-t')$$

fluctuation-dissipation theorem relates diffusion w/ noise!

We could read off the form of $\mathfrak{Z}_J$ by recalling that quadratic terms in MSR Lagrangian were noise variance!

Finally, let's justify that the EFT we wrote down is the right one. In particular, why did we neglect nonlinearities and higher-derivative terms?

Renormalization group (scaling analysis):

most general $\Phi = \int d^dx \left[\frac{\rho^2}{2\chi} + \frac{\alpha}{3}\rho^3 + \cdots + \kappa(\partial_x\rho)^2 + \cdots \right]$

Suppose $x \to \frac{1}{\lambda}x$ and $\lambda \to 0$ (look on long length scales)

Postulate: $\rho \to \lambda^{[\rho]}\rho$ where $[\rho]$ is scaling dimension

Then $\Phi \to \int \frac{d^dx}{\lambda^d}\left[\underbrace{\lambda^{2[\rho]}\frac{\rho^2}{2\chi}} + \cdots + \lambda^{2+2[\rho]}\kappa(\partial_x\rho)^2 + \cdots \right]$

Goal: as $\lambda \to 0$, Φ invariant. Keep only smallest exponent in λ!

Fix: $2[\rho] - d = 0$ or $[\rho] = d/2$.

$$\mathcal{L} = \pi\partial_t\rho + i\sigma(\rho)\partial_x\pi\,\partial_x(\pi - i\mu) + i\mathcal{J}(\rho)\partial_x^2\pi\,\partial_x^2(\pi - i\mu) + \cdots ?$$

Now $\pi \to \lambda^{[\pi]}\pi$, $t \to \lambda^z t$ as well.

Time-reversal: $[\pi] = [\mu]$, and $[\mu] = [\rho] = d/2$.

$[\rho] > 0 \longrightarrow$ leading order terms $\sigma(\rho) \approx \sigma_0$, etc...

Fix z: $0 = \left[\int dt\, d^dx\; i\sigma_0(\partial_x\pi)^2 \right] = -z - d + 2(1 + [\pi])$
$= -z + 2$

So $z = 2$. This is known as diffusive scaling.

The diffusive scaling that we've seen matches the dispersion relation we found from hydro EFT which also makes sense.

Since $[\partial_x]$, $[\pi]$, $[\rho] > 0$:

- any correction w/ $(\partial_x \pi)^3$ is irrelevant

  comes w/ positive power of λ

- nonlinearities in ρ irrelevant
- higher derivatives irrelevant

So our theory of diffusion is universal, capturing all leading-order behavior allowed by symmetry on sufficiently long wavelengths. Indeed:

Regime of validity for hydro EFT is always long time and length scales