PHYS 7810
Hydrodynamics
Spring 2026

Lecture 6
Quasihydrodynamics and correlation functions

January 27

Let's recap the hydro EFT of diffusion from lecture 5.

Assuming density ρ is only slow DOF, and $\Phi = \int d^dx \frac{\rho^2}{2\chi} + \cdots$

$$\mathcal{L} = \pi \partial_t \rho + i\sigma \partial_i \pi \partial_i (\pi - i\mu_i)$$

↳ noise-free EOM: $\left.\frac{\delta S}{\delta \pi}\right|_{\pi=0} = 0 = \partial_t \rho - \partial_i\left(\sigma \partial_i \frac{\rho}{\chi}\right) = \partial_t \rho - \partial_i (D\partial_i \rho)$

This derivation assumed we have T. What if we relax this to only PT symmetry (generalized T)?

Reduce to PT symmetry? $x \to -x$, $t \to -t$, $\rho \to \rho$, $\pi \to -\pi + i\mu$

Notice that we can add a new term to $\mathcal{L}$:

In $d=1$: $\mathcal{L}_{PT} = \partial_x \pi \cdot f(\mu)$ ← arbitrary function

$$\to (-\partial_x(-\pi + i\mu)) f(\mu) = \partial_x \pi \cdot f - \underbrace{i f(\mu) \partial_x \mu}_{= -i\partial_x V(\mu) \text{ if } V' = f.} \text{ under T}$$

And we can freely add total derivatives to a Lagrangian as long as we can ignore boundary conditions.

$S \to S - i\int dt\,dx\,\cancel{\partial_x V(\mu)}^{\,0}$ w/ periodic BCs e.g.

On HW2 you'll be asked to think about this point more carefully so I won't say too much more in class.

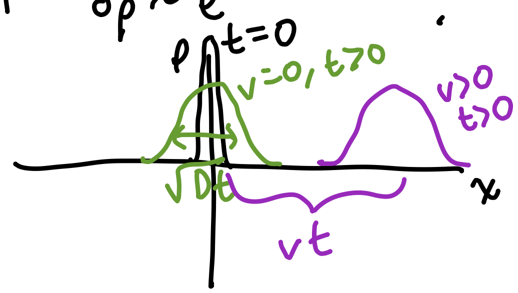
Quasinormal modes: if $\rho \to \rho_0 + \delta\rho$ and $f'(\rho_0) = -v$

$$\partial_t \rho - \partial_x f(\mu) - \partial_x(\sigma \partial_x \mu) = 0$$

$\leftarrow$ from varying $\mathcal{L}$ w/ π and setting $\pi=0$, as in lecture 5

$$\to \partial_t \delta\rho + v\,\partial_x \delta\rho - D\partial_x^2 \delta\rho = 0$$

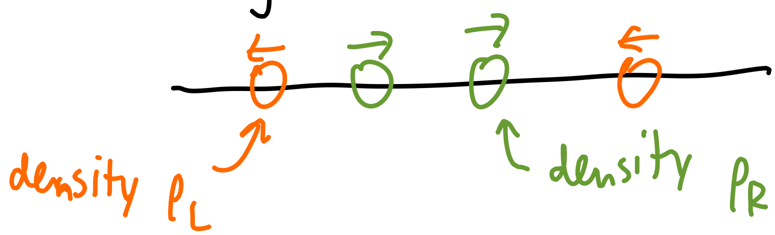
If $\delta\rho \sim e^{ikx - i\omega t}$: $\boxed{\omega = vk - iDk^2.}$ biased diffusion.



We used scaling arguments (RG) to justify that we didn't need to include higher derivative terms in ρ etc. Why not more DOF?

Example: colliding left- & right-moving particles in $d=1$.

\# of space dimensions



Suppose that particles move w/ same speed but collisions can relax relative discrepancy btwn ρ_L & ρ_R. We can take two copies of the biased diffusion theory...:

$$\mathcal{L} = \left[\pi_L \partial_t \rho_L + \partial_x \pi_L \cdot v\rho_L + i\partial_x \pi_L \partial_x(\pi_L - i\mu_L)\right] + \left[\pi_R \partial_t \rho_R - \partial_x \pi_R\, v\rho_R + i\partial_x \pi_R \partial_x(\pi_R - i\mu_R)\right]$$

and add a new term that is consistent w/:

conserved: $N = \int dx\,(\rho_L + \rho_R)$ not conserved: $N_{rel} = \int dx\,(\rho_L - \rho_R)$.

N conservation: invariance under $\begin{pmatrix}\pi_L \\ \pi_R\end{pmatrix} \to \begin{pmatrix}\pi_L \\ \pi_R\end{pmatrix} + \begin{pmatrix}1 \\ 1\end{pmatrix}$

So $\mathcal{L} \to \mathcal{L} + i\alpha\chi(\pi_L - \pi_R)(\pi_L - \pi_R - i\mu_L + i\mu_R)$ allowed.

EOMs: assume $D = \sigma/\chi$, $\mu_{L,R} = \dfrac{\rho_{L,R}}{\chi}$ as before:

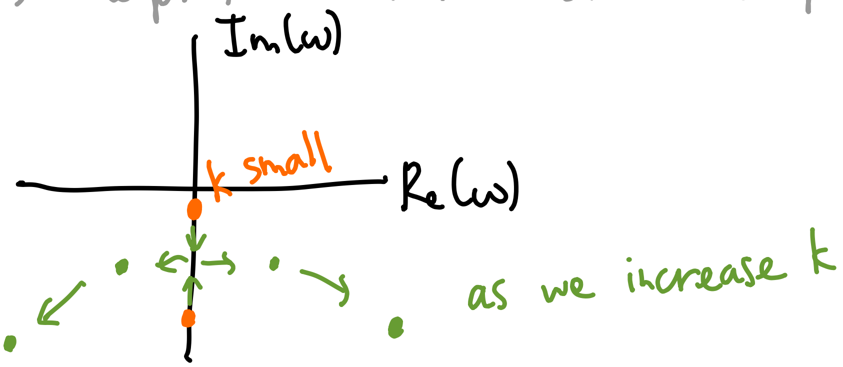
$$\partial_t \rho_L - v\partial_x \rho_L - D\partial_x^2 \rho_L = -\alpha(\rho_L - \rho_R)$$
$$\partial_t \rho_R + v\partial_x \rho_R - D\partial_x^2 \rho_R = -\alpha(\rho_R - \rho_L)$$

Quasinormal modes: $$-i\omega \begin{pmatrix} \rho_L \\ \rho_R \end{pmatrix} = \begin{pmatrix} ikv - Dk^2 - \alpha & \alpha \\ \alpha & -ikv - Dk^2 - \alpha \end{pmatrix}$$

So we need to find the eigenvalues of this matrix:

$$-i\omega = -\alpha - Dk^2 \pm \sqrt{(ikv)^2 + \alpha^2}$$
$$\omega = -i\alpha - iDk^2 \pm \sqrt{k^2v^2 - \alpha^2}$$

It's helpful to sketch how these poles move in the complex plane



As $k \to 0$: $$\omega = -i\alpha - iDk^2 \pm i\alpha\left[1 - \frac{1}{2}\left(\frac{kv}{\alpha}\right)^2 - \frac{1}{8}\left(\frac{kv}{\alpha}\right)^4 + \cdots\right]$$
$$\approx \begin{cases} \boxed{-i\left[D + \frac{v^2}{2\alpha}\right]k^2 + \cdots} \\ \boxed{-i\alpha + \cdots} \end{cases}$$

hydro mode: $\mathrm{Im}(\omega) \to 0$ as $k \to 0$

non-hydro mode $\mathrm{Im}(\omega) > 0$ as $k \to 0$

Hydro is EFT of dynamics on long time scales so we should only keep the diffusion mode w/ effective diffusion constant

$D_{eff} = D + \dfrac{v^2}{2\alpha}$ $\longrightarrow$ captured by old EFT w/ just ρ

This cartoon hopefully helps convince you that the hydro postulate on what the DOF are is reasonable.

Hydrodynamic EFT breaks down at $\underbrace{\text{short time/length scales}}_{\text{when } \Delta t \lesssim \frac{1}{\alpha} \text{ or } \Delta x \lesssim \ell = \frac{v}{\alpha}}$!

mean free path

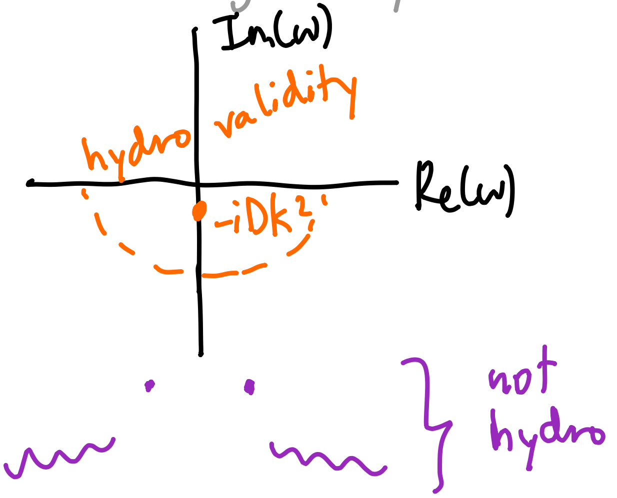
Indeed: if $\alpha \to 0$ so $D_{eff} \approx \frac{v^2}{2\alpha}$:

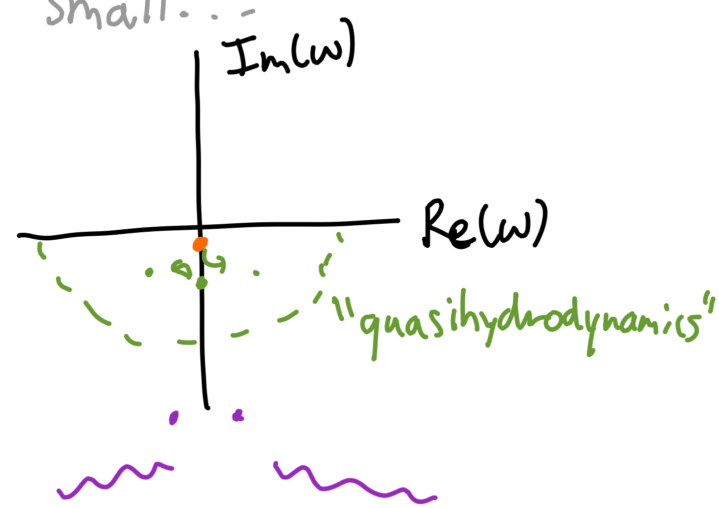
$$\omega = -i D_{eff} k^2 \left[1 + \left(\frac{k\ell}{2}\right)^2 + \dots\right]$$

and derivative corrections to hydro EFT are relevant on scales comparable to the mean free path.

Hydro regime of validity: $\omega/\alpha \ll 1$ & $k\ell \ll 1$

Sometimes it's useful to keep track of an almost-conserved DOF: e.g. maybe α is very small...





Quasihydro = keeping track of a small # of "almost conserved" quantities, or adding weak relaxation mechanisms to otherwise exactly conserved ones. Can use similar tools to study this...

Experimental consequences of hydro? $\Longrightarrow$ correlation functions

Symmetric Green's func.   retarded Green's func:

$$G^S_{AA}(x,t) = \frac{1}{2}\langle \{A(x,t), A(0,0)\}\rangle \qquad G^R_{AA} = i\Theta(t)\langle [A(x,t), A]\rangle$$

Some generic op.   quantum operators   avg in Φ: steady state

Here we're assuming space & time translation symmetry.

Fourier transform: $G^S_{AA}(k,\omega) = \langle \frac{1}{2}\{A(-k,-\omega), A(k,\omega)\}\rangle$

Claim: if A "overlaps" ρ, we see hydrodynamic poles.

e.g. $A=\rho$. let's use MSR path integral to calculate...

Classically, "operators commute" so G^S natural:

$$G^S_{\rho\rho}(x,t) \sim \int D\rho\, D\pi\; \underbrace{e^{iS_{MSR}}\cdot \rho(x,t)\,\rho(0,0)}_{\text{Gaussian (path) integral!}}$$

similar to calculating correlation functions in QFT.

$$\Big\downarrow$$

$$G^S_{\rho\rho}(k,\omega) \sim \int D\rho\, D\pi\;\; \rho(k,\omega)\rho(-k,-\omega)\cdot \exp\left[\frac{i}{2}\int dk\, d\omega\, (\rho\;\; \pi)_{k,-\omega} M_{k,\omega}\begin{pmatrix}\rho\\ \pi\end{pmatrix}_{k,\omega}\right]$$

for simplicity: T-symmetric, no nonlinearity

use $\mathcal{L}$ to deduce M

$$\mathcal{L} = \pi\partial_t\rho + i\sigma\,\partial_x\pi\,\partial_x(\pi - i\rho/\chi) \qquad \text{in } (x,t)$$

Fourier

$$S=\int dt\,dx\,\mathcal{L} = \int dk\,d\omega\left[\pi(-k,-\omega)(-i\omega)\rho(k,\omega) + i\sigma k^2\pi(-k,-\omega)\pi(k,\omega) + Dk^2\pi(-k,-\omega)\rho(k,\omega)\right] \qquad (D=\sigma/\chi)$$

Note: $\int dk\,d\omega\left[\pi(-k,-\omega)(-i\omega)\rho(k,\omega)\right] = \frac{1}{2}\int dk\,d\omega\left[\pi(-k,-\omega)(-i\omega)\rho(k,\omega) + \pi(k,\omega)(i\omega)\rho(-k,-\omega)\right]$

and we can do this "symmetrization" to read off M:

$$M_{k,\omega} = \begin{pmatrix} 0 & i\omega + Dk^2 \\ -i\omega + Dk^2 & 2i\sigma k^2\end{pmatrix}$$

Now let's quote a result about Gaussian path integrals:

$$\langle \rho(-k,-\omega)\rho(k,\omega)\rangle = \left[\left(-iM_{k,\omega}\right)^{-1}\right]_{\underset{\text{matrix component}}{\rho\rho}}, \quad \langle \rho(-k,-\omega)\pi(k,\omega)\rangle = \left[(-iM)^{-1}\right]_{\rho\pi}$$

$$(-iM)^{-1} = \begin{pmatrix} \frac{2\sigma k^2}{\omega^2+(Dk^2)^2} & \frac{-1}{\omega+iDk^2} \\ \frac{1}{\omega-iDk^2} & 0 \end{pmatrix} \rightarrow G^S_{\rho\rho}(k,\omega) = \frac{2\sigma k^2}{\omega^2+(Dk^2)^2}$$

Note: G^S singular as $k,\omega\to 0$: $\lim_{k\to 0}\lim_{\omega\to 0} G^S_{\rho\rho} \neq \lim_{\omega\to 0}\lim_{k\to 0} G^S_{\rho\rho}$.

G^S captures statistical fluctuations in equilibrium. But G^R is more physical:

Claim: $G^R_{\rho\rho}(k,\omega) \sim \langle \rho(-k,-\omega)\pi(k,\omega)\rangle \sim \underbrace{\frac{-1}{\omega+iDk^2}}$

hydrodynamic pole at $\omega=-iDk^2$, associated to diffusion QNM

Heuristic: $G^R_{\rho\rho}(x,t) = \frac{\delta}{\delta\tilde{\mu}(0,0)}\langle\rho(x,t)\rangle_{\text{initial ensemble}}\ \underbrace{\Phi - \int dx\,\tilde{\mu}\rho}_{\mu\to\mu-\tilde{\mu}(x,t).}$

Fourier transform: $\langle\rho\rangle = \int D\pi D\rho \cdot e^{iS_{MSR}}\rho$

$$S_{MSR} \to S_{MSR} + \int dk\,d\omega\ i\sigma k^2\pi\tilde{\mu}$$

so $\frac{\partial\langle\rho\rangle}{\partial\tilde{\mu}} = i\sigma k^2\langle\rho\pi\rangle = \boxed{\frac{\sigma k^2}{Dk^2-i\omega} = G^R_{\rho\rho}(k,\omega)}$

exact answer

We're a bit sloppy about deriving prefactor in MSR but for this class we won't stress this point further.