PHYS 7810
Hydrodynamics
Spring 2026

Lecture 7
Navier-Stokes equations

January 29

Hydrodynamics of liquids & gases (fluids).
↳ dissipative EFT. DoF = conserved quantities

d spatial dimensions

particle number / mass : $N = \int d^d x \, \rho$

momentum : $P_i = \int d^d x \, g_i$

energy : $E = \int d^d x \, \varepsilon$

angular momentum : L_{ij} (discuss in lec 8/HW2)

Before building an MSR Lagrangian it will help to review the thermodynamics that arises in systems w/ conserved N, P_i, E

Thermodyamics of fluid ($k_B = 1$):

1st Law: $$T\,dS = dE - \tilde{\mu}\,dN - v_i\,dP_i + \underbrace{P\,dV}_{\text{work done by system on environment}}$$

T: temperature; $\tilde{\mu}$: chemical potential; v_i: fluid velocity; P: pressure

It's convenient to divide by volume so we're working w/ densities:

Holding volume fixed: $dV=0$:

$$ds = \frac{1}{T}d\varepsilon - \frac{\tilde{\mu}}{T}d\rho - \frac{v_i}{T}dg_i \Longrightarrow \frac{1}{T} = \left.\frac{\partial s}{\partial \varepsilon}\right|_{\rho, g_i}$$

temperature = thermodynamic conjugate of energy

Gibbs–Duhem relation: divide by dV: $\varepsilon = \frac{dE}{dV}$ etc...

$$\varepsilon + P = Ts + \tilde{\mu}\rho + v_i g_i$$

These relations will be useful to keep in mind for the lecture.
In particular, let's think about how we'll build MSR Lagrangian...

steady state: $e^{-\Phi} \longrightarrow \underbrace{e^{\int d^d x \, s(\rho, \varepsilon, g_i)}}_{\text{microcanonical ensemble, energy is conserved!}}$

So: $\mu_\rho = \frac{\delta \Phi}{\delta \rho(x)} = -\left.\frac{\partial s}{\partial \rho}\right|_{\varepsilon, g} = -\left(-\frac{\tilde{\mu}}{T}\right) = \frac{\tilde{\mu}}{T}$

Similarly: $\mu_\varepsilon = -\frac{1}{T}$, $\mu_{g_i} = \mu_i = \frac{v_i}{T}$

Time-reversal symmetry: $\rho \to \rho$ $\quad \varepsilon \to \varepsilon$ $\quad g_i \to -g_i$

$\pi_\rho \to -\pi_\rho + i\mu_\rho$ $\quad \pi_\varepsilon \to -\pi_\varepsilon + i\mu_\varepsilon$ $\quad \pi_i \to \pi_i - i\mu_i$

$$\mathcal{L}_{MSR} = \pi_\rho \partial_t \rho + \pi_\varepsilon \partial_t \varepsilon + \pi_i \partial_t g_i - \underbrace{H_{MSR}(\cdots)}_{}$$

conservation laws imply

$$\mathcal{H}_{MSR} = (\partial_i \pi_\rho) J_i + (\partial_i \pi_\varepsilon) \mathcal{E}_i + (\partial_i \pi_j) \tau_{ij}$$

particle current $\quad$ energy current $\quad$ stress tensor

so that: $\frac{\delta S}{\delta \pi_\rho} = \partial_t \rho - \partial_i(-J_i) = \partial_t \rho + \partial_i J_i = 0$ (local conservation law)

Goal: find constitutive relations for $J(\rho, \varepsilon, g_i)$?

Ideal hydro: assume $J, \mathcal{E}, \tau$ have no $\partial_i \rho$, etc.

In case of pure diffusion this was only possible when we had PT-sym only. But now we have some T-even and some T-odd DOF so we should expect this possibility from the outset.

Note: under T-sym: $J_i \to -J_i$, $\mathcal{E}_i \to -\mathcal{E}_i$, $\tau_{ij} \to \tau_{ij}$

$\tilde{\mu} \to \tilde{\mu}$ $\quad T \to T$ $\quad v_i \to -v_i$

It will be useful to express things in terms of the conjugate variables, rather than densities.

With rotation invariance:

$$\mathcal{E}_i = A(\tilde{\mu}, T, v_j v_j) v_i , \quad J_i = B v_i , \quad \tau_{ij} = C\delta_{ij} + D v_i v_j$$

T-"invariance" of MSR Lagrangian:

$$\mathcal{L}_{MSR} \to \mathcal{L}_{MSR} + i\Big[\underbrace{\mu_\rho \partial_t \rho + \mu_\varepsilon \partial_t \varepsilon + \mu_i \partial_t g_i}_{-\partial_t s:\ \text{detailed balance!}} - \underbrace{J_i \partial_i \mu_\rho - \mathcal{E}_i \partial_i \mu_\varepsilon - \tau_{ij}\partial_i \mu_j}_{\text{total divergence? } -\partial_i \tilde{S}_i}\Big]$$

$$J_i \partial_i \mu_\rho + \cdots = B v_i \partial_i\left(\frac{\tilde{\mu}}{T}\right) + A v_i \partial_i \left(\frac{-1}{T}\right) + \underbrace{C \partial_i \left(\frac{v_i}{T}\right)}_{= \partial_i\left(C\frac{v_i}{T}\right) - \frac{v_i}{T}\partial_i C} + D v_i v_j \partial_i \left(\frac{v_j}{T}\right) = -\partial_i \tilde{S}_i ?$$

Write: $C\left(\frac{1}{T}, \frac{\tilde{\mu}}{T}, \frac{1}{2T} v_j v_j\right)$:

$$\partial_i C = \left.\frac{\partial C}{\partial\left(\frac{1}{T}\right)}\right|_{\frac{\tilde{\mu}}{T}, \frac{v^2}{2T}} \partial_i \frac{1}{T} + \left.\frac{\partial C}{\partial\left(\frac{\tilde{\mu}}{T}\right)}\right|_{\frac{1}{T}, \frac{v^2}{2T}} \partial_i \left(\frac{\tilde{\mu}}{T}\right) + \left.\frac{\partial C}{\partial\left(\frac{v^2}{2T}\right)}\right|_{\frac{1}{T}, \tilde{\mu}} \underbrace{\partial_i \left(\frac{v^2}{2T}\right)}_{= -\frac{v^2}{2}\partial_i \frac{1}{T} + v_j \partial_i \left(\frac{v_j}{T}\right)}$$

Collect coefficients of...

$$\partial_i\left(\frac{v_j}{T}\right): \quad D v_i v_j - v_j \frac{v_i}{T}\frac{\partial C}{\partial\left(\frac{v^2}{2T}\right)}\Bigg|_{\frac{1}{T},\tilde{\mu}} = 0 = v_i v_j\left[D - \frac{\partial C}{\partial\left(\frac{v^2}{2}\right)}\right]$$

$$\text{so} \quad D = \frac{\partial C}{\partial\left(\frac{v^2}{2}\right)}$$

$$\partial_i\left(\frac{\tilde{\mu}}{T}\right): \quad B v_i - \frac{v_i}{T}\frac{\partial C}{\partial(\tilde{\mu}/T)}\Bigg|_{\frac{1}{T},\frac{v^2}{2T}} = 0 \quad \Rightarrow \quad B = \frac{\partial C}{\partial\tilde{\mu}}$$

A more annoying calculation leads to A. But we can give a nice expression for A if we demand that

Claim: $C \to$ pressure P (which we'll justify at end of lecture)

Then: $B = \frac{\partial P}{\partial\tilde{\mu}} = \rho$ and $A = \varepsilon + P$

$$\hookrightarrow \text{since } dP = d\left[\rho\tilde{\mu} + Ts + v_i g_i - \varepsilon\right] = \rho\, d\tilde{\mu} + s\, dT + g_i\, dv_i$$

Similarly: $\frac{\partial P}{\partial v_i} = v_i \frac{\partial P}{\partial\left(v^2/2\right)} = D v_i$ so $D \to$ "mass density" (lec 8)

To summarize, constitutive relations for ideal fluid are:

$$J_i = \rho v_i, \quad \mathcal{E}_i = (\varepsilon + P)v_i, \quad \tau_{ij} = P\delta_{ij} + \frac{\partial P}{\partial\left(\frac{v^2}{2}\right)} v_i v_j$$

Now let's add dissipative corrections, again assuming T and rotational symmetry. As in diffusion, we should add as few derivatives as possible b/c hydro is long-wavelength EFT. The most general possible terms to add are:

$$\mathcal{L}_{MSR,diss} = iT\eta\, \partial\pi_i\, \partial(\pi - i\mu)_i + iT\mathcal{A}\, \partial\pi_{\varepsilon/\rho}\, \partial(\pi - i\mu)_{\varepsilon/\rho} + iT\mathcal{B}v_i\left[\partial\pi_j\, \partial(\pi - i\mu)_{\varepsilon/\rho} + \partial(\pi - i\mu)_j\, \partial\pi_{\varepsilon/\rho}\right]$$

schematic

One can check all of these terms are T-invariant.
To be more explicit, worry about indices...

momentum dissipation: viscosity tensor η_{ijkl}:

$$i T \eta_{ijkl} \partial_i \pi_j \partial_k(\pi_l - i\mu_l)$$

assume rotational symmetry

$$\eta_{ijkl} = \eta\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj} - \frac{2}{d}\delta_{ij}\delta_{kl}\right) + \zeta\, \delta_{ij}\delta_{kl}$$

shear viscosity

bulk viscosity

Here η & ζ are phenomenological parameters of the EFT.

$$\mathrm{Im}(\mathcal{H}_{MSR}) \le 0 \quad \Rightarrow \quad \eta, \zeta \ge 0$$

particle num / energy dissipation: (incoherent) thermoelectric conductivity.

$$i T A\, \partial\pi\, \partial(\pi - i\mu) \rightarrow i T \begin{pmatrix} \partial_i \pi_\rho & \partial_i \pi_\varepsilon \end{pmatrix} \begin{pmatrix} \sigma_0 & T\alpha \\ T\alpha & T\bar{\kappa} \end{pmatrix} \begin{pmatrix} \partial_i(\pi_\rho - i\mu_\rho) \\ \partial_i(\pi_\varepsilon - i\mu_\varepsilon) \end{pmatrix}$$

similar positivity conditions...

Mixed term:

$$\sum_{a=\varepsilon,\rho} i T B^a_{ijkl} v_i \left[\partial_j \pi_a \partial_k(\pi_l - i\mu_l) + \partial_j(\pi_a - i\mu_a)\partial_k \pi_l\right]$$

$$B_{ijkl} = B_1 \delta_{ij}\delta_{kl} + B_2 \delta_{ik}\delta_{jl} + B_3 \delta_{il}\delta_{jk}$$

Now we can write down equations of motion by varying with respect to π, as we did for diffusion.

Navier-Stokes equations: (ignore B for simplicity)

$$\partial_t \rho + \partial_i J_i = 0 \qquad \text{where} \quad J_i = \rho v_i - T\sigma_0 \partial_i \frac{\tilde{\mu}}{T} - \alpha \partial_i T$$

$$\partial_t \varepsilon + \partial_i \mathcal{E}_i = 0 \qquad \text{where} \quad \mathcal{E}_i = (\varepsilon + P) v_i - T^2 \alpha\, \partial_i \frac{\tilde{\mu}}{T} - \bar{\kappa}\, \partial_i T$$

Onsager reciprocity: same coefficients, consequence of T-inv.

$$\partial_t g_i + \partial_j \tau_{ij} = 0 \qquad \text{where} \quad \tau_{ij} = P\delta_{ij} + \frac{\partial P}{\partial(\frac{v^2}{2})} v_i v_j - T\eta_{ijkl}\, \partial_k \frac{v_l}{T}$$

The next third of the class will be spent analyzing these equations and understanding the physics they imply.

The last thing for today is to understand the origin of the identification of pressure. To do so:

Claim: our hydro is consistent w/ 2nd law of thermo!

↳ in abstract shorthand:

$$\mathcal{L}_{MSR} = \underset{\substack{\uparrow \\ \alpha \text{ runs over } \varepsilon, \rho, g_i}}{\pi_\alpha} \partial_t \rho_\alpha - \partial_i \pi_\alpha \underset{\hookrightarrow \{J_i, \mathcal{E}_i, \tau_{ij}\}}{R_{i\alpha}} + i\partial_i \pi_\alpha \Sigma_{ij\alpha\beta} \partial_j (\pi_\beta - i\mu_\beta)$$

Recall: $\mu_\alpha \partial_t \rho_\alpha = -\partial_t s$ ← entropy density.

Hydro EOMs: (w/o noise)

$$0 = \partial_t \rho_\alpha + \partial_i R_{i\alpha} - \partial_i (\Sigma_{ij\alpha\beta} \partial_j \mu_\beta)$$

↳ multiply by μ_α:

$$\partial_t s = \underbrace{\partial_i(\mu_\alpha R_{i\alpha}) - R_{i\alpha} \partial_i \mu_\alpha}_{\text{from before: } \partial_i \left(-\frac{\varepsilon+P}{T} v_i + \frac{\tilde{\mu}\rho}{T} v_i + \frac{v_i P}{T} + \frac{v_j}{T} g_j v_i \right) - \partial_i \left(\frac{P}{T} v_i \right) \,=\, -\partial_i (s v_i) \text{ using Gibbs-Duhem!}} + \mu_\alpha \partial_i (\Sigma_{ij\alpha\beta} \partial_j \mu_\beta)$$

if C=P

Define entropy current: $\mathcal{S}_i = s v_i + \mu_\alpha \Sigma_{ij\alpha\beta} \partial_j \mu_\beta$:

$$\partial_t s + \partial_i \mathcal{S}_i = \underbrace{\Sigma_{ij\alpha\beta} \partial_i \mu_\alpha \partial_j \mu_\beta}_{\geq 0 \,:\, 2^{nd} \text{ law of thermo (w/o noise)}}$$

≥0 ← implied by $\mathrm{Im}(\mathcal{H}_{MSR}) \leq 0$