PHYS 7810
Hydrodynamics
Spring 2026

Lecture 8
Galilean invariance

February 3

Recap: liquids & gases have conserved:

mass: $\int d^d x \; \rho$

momentum: $\int d^d x \; g_i$

energy: $\int d^d x \; \varepsilon$

conjugate to $\longrightarrow$

$\mu_\rho = \tilde{\mu}/T$

$\mu_i = v_i/T$ ← fluid velocity

$\mu_\varepsilon = -1/T$

Last time we wrote down MSR Lagrangians consistent w/ these symmetries. Today we will impose additional symmetries suitable for liquids & gases in our everyday non-relativistic world.

Rotational invariance $\longrightarrow$ angular momentum conservation.

On HW 2 you'll explore this in some detail, here summarize...

since $\partial_t g_i + \partial_j \tau_{ji} = 0$.

$$\frac{d}{dt}\int d^d x \left[x_i g_j - x_j g_i\right] = \int d^d x \left[-x_i \partial_k \tau_{kj} + x_j \partial_k \tau_{ki}\right]$$

local angular momentum density

$$= \int d^d x \left[\delta_{ik}\tau_{kj} - \delta_{kj}\tau_{ki}\right] = \int d^d x \left(\tau_{ij} - \tau_{ji}\right)$$

We take $\tau_{ij} = \tau_{ji}$ as a consequence of rotational sym.

The second important symmetry to consider is Galilean invariance.
Consider a microscopic model where all particles are non-relativistic and have $\frac{m}{2}v^2$ kinetic energy etc...

$$P_i = \int d^d x \, g_i \rightarrow \sum_{\text{particle } \alpha} p_{i,\alpha} = \sum_\alpha m \dot{x}_{i,\alpha}$$

(assume all particles have same mass)

local expression for current, so $g_i = J_i$

This can be derived more carefully based on analyzing the Galilean symmetry algebra, the details of which are beyond class.

Recall that at ideal order, $g_i = \underbrace{\frac{\partial P}{\partial\left(\frac{v^2}{2}\right)}}_{\text{call this } \tilde{\rho}.} v_i$. $\quad J_i = \frac{\partial P}{\partial \tilde{\mu}} v_i = \rho v_i$.

Therefore $\tilde{\rho} = \rho$, so pressure $P(T, \tilde{\mu} + v^2/2)$

Helpful to interpret: $P \rightarrow \underbrace{P_0(T, \rho)}_{\text{physics same in all ref. frames}}$ (different function, but only depends on T, ρ)

Gibbs-Duhem:
$$\begin{aligned} \varepsilon + P &= \tilde{\mu}\frac{\partial P}{\partial \tilde{\mu}} + T\frac{\partial P}{\partial T} + \frac{v^2}{2}\frac{\partial P}{\partial v^2/2} \\ &= \left(\tilde{\mu} + \frac{v^2}{2}\right)\rho + T\frac{\partial P}{\partial T} = \underbrace{\varepsilon_0 + P_0}_{\text{energy density, pressure at } v=0} + \rho\frac{v^2}{2} \end{aligned}$$

So let's summarize what we've learned about ideal hydro:

$$\partial_t \rho + \partial_i J_i = 0 \qquad J_i = \rho v_i = g_i$$

$$\partial_t g_i + \partial_j \tau_{ji} = 0 \qquad \tau_{ij} = \frac{\partial P}{\partial\left(\frac{v^2}{2}\right)} v_i v_j + P\delta_{ij} = \rho v_i v_j + P\delta_{ij}$$

$$\partial_t \varepsilon + \partial_i \mathcal{E}_i = 0 \qquad \mathcal{E}_i = (\varepsilon_0 + P_0)v_i + \frac{\rho}{2}v^2 v_i$$

Now let's return to the MSR formalism. What dissipative terms can we write down?

Rotation symmetry: only couple to $\partial_i \pi_j + \partial_j \pi_i$ (see HW 2)

Galilean symmetry: $J_i = \rho v_i$ means no π_ρ $(\pi_\rho - i\mu_\rho)$ dissipation!

↳ and since $v_i \to v_i + u_i$ under boost, couple only to $\partial_i v_j$ in dissipative terms.

$\partial_i \mu_j = \frac{v_j}{T} \to \partial_i \mu_j + \partial_i \frac{u_j}{T} = \partial_i \mu_j - u_j \partial_i \mu_\varepsilon$ under boost.

so $\partial_i \mu_j + v_j \partial_i \mu_\varepsilon = \frac{1}{T} \partial_i v_j$ is invariant under boost.

↳ dissipative corrections depend on $\pi_j + v_j \pi_\varepsilon$!

So w/ rotational invariance there are 3 dissipative motifs we can write down in MSR formalism:

$$\mathcal{L}_{diss} \subset i T \eta_{ijkl} \partial_i [\pi_j + v_j \pi_\varepsilon] \partial_k [\pi_l + v_j \pi_\varepsilon - \frac{i}{T} v_l]$$

$$= \eta_{klij} = \underbrace{\eta}_{\text{shear viscosity}} \underbrace{[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{d}\delta_{ij}\delta_{kl}]}_{\text{projects onto traceless symmetric tensor}} + \underbrace{\zeta}_{\text{bulk viscosity}} \delta_{ij}\delta_{kl}$$

$$\mathcal{L}_{diss} \subset i \underbrace{\kappa}_{\text{thermal conductivity}} T^2 \partial_i \pi_\varepsilon \partial_i (\pi_\varepsilon - i\mu_\varepsilon)$$

Combine w/ ideal hydro to write Navier-Stokes equations:

$$\partial_t \rho + \partial_i (\rho v_i) = 0$$

$$\partial_t (\rho v_i) + \partial_j \left[P \delta_{ij} + \rho v_i v_j - \eta(\partial_i v_j + \partial_j v_i) - (\zeta - \tfrac{2\eta}{d}) \partial_k v_k \delta_{ij} \right] = 0$$

$$\partial_t \varepsilon + \partial_i \left[(\varepsilon_0 + P_0 + \tfrac{\rho v^2}{2}) v_i - v_j [\eta(\partial_i v_j + \partial_j v_i) + (\zeta - \tfrac{2\eta}{d}) \delta_{ij} \partial_k v_k] - \kappa \partial_i T \right] = 0$$

As usual we neglect noise when writing down dissipative EOMs

The advantage of MSR is that every detail of Galilean invariance is manifest in our derivation!

Re-write momentum equation using mass equation:

$$\rho\underbrace{\left[\partial_t v_i + v_j\partial_j v_i\right]}_{\text{convective derivative: } d_t = \partial_t + \vec{v}\cdot\nabla} + \partial_i P - [\text{viscous terms}] = 0.$$

As in the case of diffusion we can now calculate the quasinormal modes of the fluid.

Linearize around equilibrium:

$$\rho = \rho_0 + \delta\rho\, e^{ikx - i\omega t}$$
$$\varepsilon = \varepsilon_0 + \delta\varepsilon\, e^{ikx - i\omega t}$$
$$v_i = 0 + \delta v_i\, e^{ikx - i\omega t}$$

Navier-Stokes equations:

$$0 = -i\omega\delta\rho + ik\rho_0\delta v_x$$
$$0 = -i\omega\delta\varepsilon + ik(\varepsilon_0 + P_0)\delta v_x + \kappa k^2\left(\left.\frac{\partial T}{\partial\rho}\right|_\varepsilon\delta\rho + \left.\frac{\partial T}{\partial\varepsilon}\right|_\rho\delta\varepsilon\right)$$

$$\text{mom.}\left\{\begin{array}{l} 0 = -i\omega\rho_0\delta v_x + ik\left(\left.\frac{\partial P}{\partial\rho}\right|_\varepsilon\delta\rho + \left.\frac{\partial P}{\partial\varepsilon}\right|_\rho\delta\varepsilon\right) + \left(\frac{2d-2}{d} + \zeta\right)k^2\delta v_x \\ 0 = -i\omega\rho_0\delta v_\perp + \eta k^2\delta v_\perp \end{array}\right.$$

Now let's try to find all the eigenvalues & eigenvectors of this set of equations.

① Transverse momentum: only $\delta v_\perp \neq 0$:

$$\omega = -i\boxed{\frac{\eta}{\rho_0}}k^2 \longrightarrow \text{define as } \nu \text{ (kinematic viscosity)}$$

We see that (shear) viscosity has the interpretation as the diffusion constant of momentum.

To find every other QNM, need to look in longitudinal sector:

$$i\omega \begin{pmatrix} \delta\rho \\ \delta\varepsilon \\ \delta v_x \end{pmatrix} = \begin{pmatrix} 0 & 0 & ik\rho_0 \\ \cancel{\kappa k^2 \partial_\rho T} & \cancel{\kappa k^2 \partial_\varepsilon T} & ik(\varepsilon_0 + P_0) \\ \frac{\partial_\rho P}{\rho_0}\cdot ik & \frac{\partial_\varepsilon P}{\rho_0}\cdot ik & \cancel{\left(\frac{2d-2}{d}\eta + \zeta\right)k^2}/\rho_0 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta\varepsilon \\ \delta v_x \end{pmatrix}$$

Start by keeping only leading order terms in k:

This makes sense b/c hydro is EFT on long wavelengths ($k \to 0$):

$$\omega \cdot \delta\rho = k\rho_0 \cdot \delta v_x \qquad \omega \cdot \delta\varepsilon = ik(\varepsilon_0 + P_0)\cdot \delta v_x$$

$$\hookrightarrow \omega \cdot \delta v_x = k \cdot \frac{k}{\omega}\left[\partial_\rho P + \frac{\varepsilon_0 + P_0}{\rho_0}\partial_\varepsilon P\right]\delta v_x$$

(2): $\omega = \pm \underbrace{v_s}_{}\, k$ speed of sound $v_s^2 = \underbrace{\partial_\rho P + \frac{\varepsilon_0 + P_0}{\rho_0}\partial_\varepsilon P}_{}$

real → not dissipative. propagating waves!

sound modes!

using thermo: $T\,ds = d\varepsilon - \tilde{\mu}\,d\rho$

Hold $\gamma = \frac{s}{\rho}$ fixed: $\gamma T\,d\rho = d\varepsilon - \tilde{\mu}\,d\rho$

$$d\varepsilon = \left(\tilde{\mu} + \frac{Ts}{\rho}\right)d\rho = \frac{\varepsilon + P}{\rho}\,d\rho \,.$$

Thus $v_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_{\rho/s}$

We can now calculate the dissipative corrections:

$$\omega = \pm v_s k - \frac{i}{2}k^2 \underbrace{\left[\frac{\kappa\rho_0}{\varepsilon_0 + P_0}\left.\frac{\partial T}{\partial\rho}\right|_{\rho/s} + \frac{1}{\rho_0}\left(\frac{2d-2}{d}\eta + \zeta\right)\right]}_{} + \cdots$$

dissipative decay: attenuation of sound waves

But we should have a third eigenvalue in the longitudinal sector.

(3) Last mode has $\omega = 0$ at $O(k)$.

Incoherent diffusion: fluctuation in $\delta\varepsilon$, $\delta\rho$ w/ $\delta P, \delta v_x \approx 0$.

Need: $\delta_\rho \approx -\frac{1}{v_s^2}\left.\frac{\partial P}{\partial \varepsilon}\right|_\rho$, $\delta_\varepsilon \approx \frac{1}{v_s^2}\left.\frac{\partial P}{\partial \rho}\right|_\varepsilon$:

$$\omega = -iDk^2$$

$$\hookrightarrow D = \frac{\kappa}{v_s^2}\left[\left.\frac{\partial P}{\partial \rho}\right|_\varepsilon \left.\frac{\partial T}{\partial \varepsilon}\right|_\rho - \left.\frac{\partial P}{\partial \varepsilon}\right|_\rho \left.\frac{\partial T}{\partial \rho}\right|_\varepsilon\right]$$

In typical liquids, $\left.\frac{\partial T}{\partial \rho}\right|_\varepsilon$ negligible so:

$$D = \underbrace{\frac{\kappa}{\left.\frac{\partial \varepsilon}{\partial T}\right|_\rho}}_{\frac{\kappa}{c}} \cdot \underbrace{\frac{\left.\partial P/\partial\rho\right|_\varepsilon}{\left.\partial P/\partial\rho\right|_{\rho/s}}}_{\approx 1}$$

specific heat $\rightarrow c$