PHYS 7810
Hydrodynamics
Spring 2026

Lecture 9
Shock and rarefaction waves

February 5

Last time, we saw that in liquids and gases, there were propagating sound modes in addition to diffusive modes:

Sound waves (near equilibrium): $\partial_t^2 \rho \approx v_s^2 \nabla^2 \rho$

Everything we know about solutions to wave equations from e.g. E&M will also apply here.

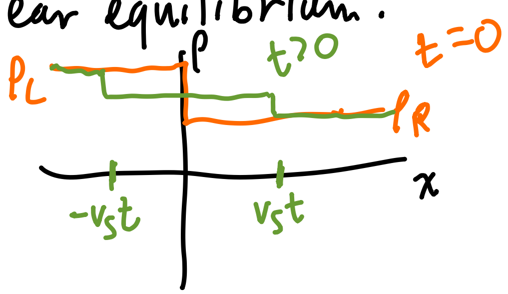
But hydro is intrinsically a nonlinear theory, so are there interesting new phenomena that arise?

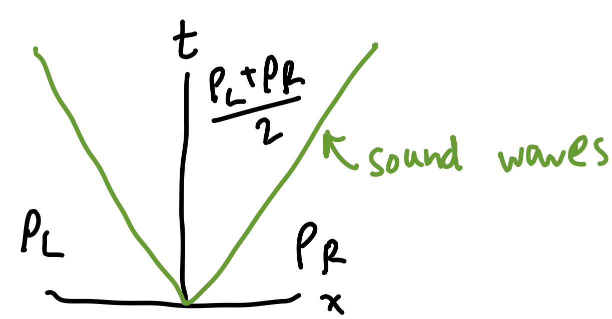
Setup: Riemann problem: $\underbrace{(\rho_L, v_{xL}, T_L) \,\big|\, (\rho_R, v_{xR}, T_R)}_{x=0}$ at $t=0$

Translation invariance in other coords & homogeneous for $x<0$ & $x>0$.

What happens at $t>0$? Assume:
- dissipationless hydro.
- neglect energy conserv.

Near equilibrium:





What does the nonlinear solution of hydro look like?
This can be relevant for gas dynamics where ρ_L, ρ_R may be quite different...

Does the discontinuity persist? $\xrightarrow{\text{yes}}$ shock wave; $\xrightarrow{\text{no}}$ rarefaction wave

Both are possible!

Let's start by analyzing a possible shock wave.

$\rightarrow v_{sh}$ = shock speed

$$(\rho_L, v_L) \mid (\rho_R, v_R) \quad x \qquad \xrightarrow{\text{boost}} \qquad (\rho_L, \underbrace{v_L - v_{sh}}_{\tilde{v}_L}) \overset{\text{static}}{\mid} (\rho_R, \overbrace{v_R - v_{sh}}^{\tilde{v}_R}) \quad x$$

$$\cancel{\partial_t \rho}^{\,\text{static}} + \partial_x(\rho v) = 0 \quad \rightarrow \quad \rho_L \tilde{v}_L = \rho_R \tilde{v}_R$$

$$\cancel{\partial_t(\rho v)} + \partial_x\left[P(\rho) + \rho v^2\right] = 0 \quad \rightarrow \quad P_L + \rho_L \tilde{v}_L^2 = P_R + \rho_R \tilde{v}_R^2$$

$$\hookrightarrow \tilde{v}_R = \frac{\rho_L}{\rho_R}\tilde{v}_L \quad \text{and} \quad \underbrace{P_L + \rho_L \tilde{v}_L^2 = P_R + \frac{\rho_L^2}{\rho_R}\tilde{v}_L^2}_{\text{Solve this nonlinear equation for } \rho_R = \rho_R^*}$$

For given $\rho_L, \tilde{v}_L \rightarrow$ fix $\rho_R \rightarrow \rho_R^*$ and $\tilde{v}_R = \frac{\rho_L}{\rho_R}\tilde{v}_R$

But this forces 2 of 4 initial data to be fixed?

$\rightarrow$ one family of shocks from tuning $v_{sh} \neq 0$.

We shouldn't have expected to fully solve Riemann w/ one shock anyway — the linear response solution had 2 "shocks"?

Before we return to full Riemann, need to consider the rarefaction possibility.

Rarefaction wave? Assume $\rho(\xi)$ and $v(\xi)$ where $\xi = \frac{x}{t}$

Notice our linear response solution also consistent w/ this ansatz!

$$\partial_t \rho + \partial_x(\rho v) = 0 \longrightarrow -\frac{x}{t^2}\partial_\xi \rho + \frac{1}{t}\partial_\xi(\rho v) = 0$$

$$-\xi \partial_\xi \rho + \partial_\xi(\rho v) = 0$$

Similarly: $$-\xi\partial_\xi(\rho v) + \partial_\xi\left[P(\rho) + \rho v^2\right] = 0$$

Combine: $$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} -\xi + v & \rho \\ P'(\rho) + v(v-\xi) & -\xi\rho + 2\rho v \end{pmatrix}}_{\text{determinant} = 0 \text{ if non-trivial solution } \begin{pmatrix} \partial_\xi \rho \\ \partial_\xi v \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.} \begin{pmatrix} \partial_\xi \rho \\ \partial_\xi v \end{pmatrix}$$

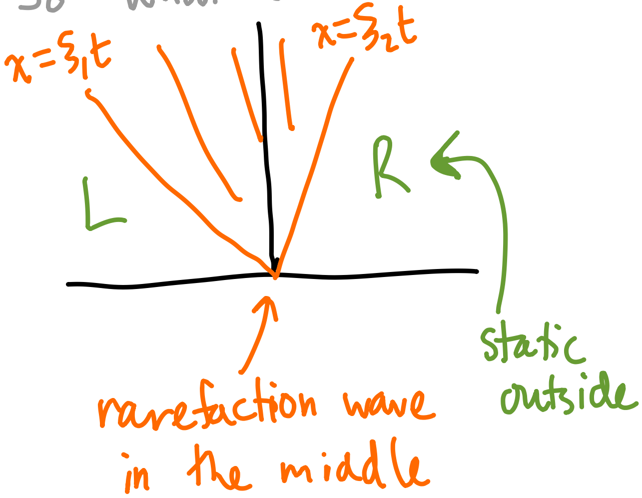
$$0 = \rho(v-\xi)(2v-\xi) - \rho\left[P'(\rho) + v(v-\xi)\right]$$

$$P'(\rho) = \underbrace{(v-\xi)^2 = v_s(\rho)^2}_{\text{solve this equation for } \rho:\ \rho_*(v-\xi)}$$ (local speed of sound!)

Then: $$0 = (v-\xi)\left(-\rho_*'(v-\xi)\right) + \left[\rho_*(v-\xi) + (v-\xi)\rho_*'(v-\xi)\right] v'(\xi)$$

↳ Solve for $v(\xi)$. Deduce $\rho(\xi)$

So what are the allowed rarefaction waves?



we need:

$\rho_L = \rho(\xi_1)$ $\qquad$ $\rho_R = \rho(\xi_2)$

$v_L = v(\xi_1)$ $\qquad$ $v_R = v(\xi_2)$

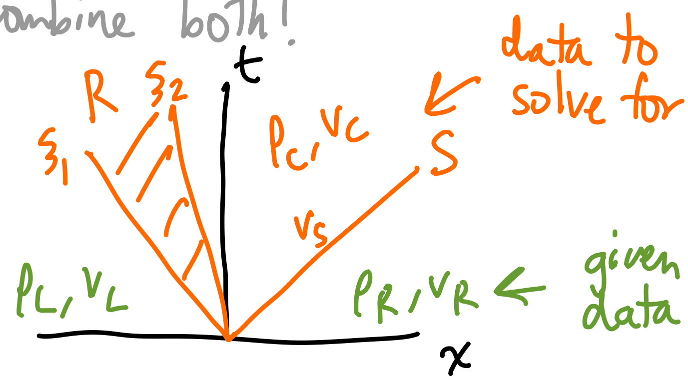
How many "degrees of freedom"?

- integration constant in $v(\xi)$
- ξ_1 & ξ_2 unknown

} potentially match 3/4 initial data

So similar to shock waves, a single rarefaction can't solve Riemann problem. Need to combine both!

Potential Riemann solution:



Crude: each R/S wave adjusts one of (ρ, v) to desired state.
May need to try RR, SR, SS solutions too...

Warning: Riemann does NOT have unique solutions

↳ fix using physics. Guess must be consistent w/ 2nd law.

rarefaction? → viscous $\rho\partial_x^2 v \sim \frac{\rho}{t}\partial_x v$ is negligible as $t\to\infty$

shock? → $\underbrace{\rho\partial_x^2 v \gg \partial_x v}_{\text{shock wave must be smoothed out by viscosity!}}$

Estimate shock width ℓ by estimating entropy production!

$$\rho\left(\frac{\Delta v}{\ell}\right)^2 \cdot \ell \sim \underbrace{T\dot{S}}_{\text{entropy production}} \sim T\big(s(\rho_c) - s(\rho_R)\big)v_{sh}$$

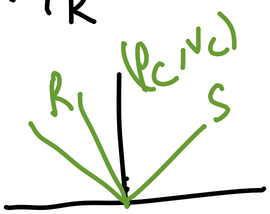
↳ $\ell \sim \frac{1}{\rho} \times (\cdots)$

Example: gas dynamics. Assume $P(\rho) = A\rho^\gamma$

A, γ constants

$$(\rho_L, v_L = 0) \,\Big|\, (\rho_R, v_R = 0)$$

$\rho_L > \rho_R$

Claim: 

with $v_c > 0$ and $\rho_L > \rho_c > \rho_R$

Let's first deduce the rarefaction wave

set $A\gamma = 1$

$$(v-\xi)^2 = P'(\rho) = A\gamma\rho^{\gamma-1} \longrightarrow \rho = \cancel{(A\gamma)^{\frac{-1}{\gamma-1}}}\,(v-\xi)^{\frac{2}{\gamma-1}}$$

$$(v-\xi)\rho' + \rho v' = 0 = \left[\frac{2}{\gamma-1}(v'-1) + v'\right](v-\xi)^{\frac{2}{\gamma-1}}$$

$$0 = \left[\frac{\gamma+1}{\gamma-1}v' - \frac{2}{\gamma-1}\right](v-\xi)^{\frac{2}{\gamma-1}}$$

So $v(\xi) = \cancelto{0}{v(\xi_1)} + \dfrac{2}{\gamma+1}(\xi - \xi_1)$.

Rarefaction starts at ξ_1: $\boxed{\rho_L = (-\xi_1)^{\frac{2}{\gamma-1}} \text{ fixes } \xi_1.}$

Don't know ξ_2 but fix ρ_c & v_c...:

$$\boxed{v_c = \frac{2}{\gamma+1}(\xi_2 - \xi_1)} \quad \text{and} \quad \boxed{\rho_c = \rho_L\left|\frac{2}{\gamma+1} + \frac{\xi_2}{\xi_1}\frac{\gamma-1}{\gamma+1}\right|^{\frac{2}{\gamma-1}}}$$

(ρ_c, v_c) expressed in terms of known ρ_L
unknown ξ_2

Now let's try to fix shock wave

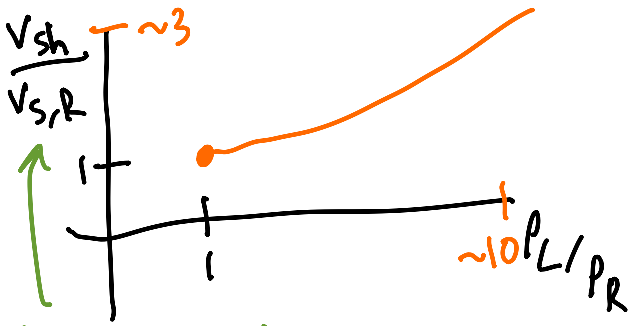
$$\rho_R(-v_{sh}) = \rho_c(v_c - v_{sh}) \longrightarrow v_{sh} = \frac{\rho_c v_c}{\rho_c - \rho_R}.$$

$$P(\rho_C) + \rho_C (v_C - v_{sh})^2 = P(\rho_R) + \rho_R v_{sh}^2$$

$$\hookrightarrow \boxed{\frac{1}{\gamma}\left[\rho_R^{\gamma} - \rho_C^{\gamma}\right] = \frac{\rho_R \rho_C}{\rho_R - \rho_C} v_C^2}$$

$\rho_C(\xi_2)$, $v_C(\xi_2) \longrightarrow$ solve nonlinear equation for ξ_2.

Numerical solution for physical $\gamma = 5/3$:



sound speed in right fluid: $\sim \rho_R^{\frac{\gamma-1}{2}}$