1D Atoms

In this problem, we will consider toy model for the energy of an atom of atomic number Z. For simplicity, we will work in a one dimensional space, and model the Coulomb interaction with a δ function interaction. Making the usual approximation that the heavy nucleus is fixed and the light electrons can move around, we have a wave function $\Psi(x_1, \ldots, x_n)$ for the electrons. For simplicity, assume that the electrons are bosons. A reasonable guess for the Hamiltonian is

$$H = H_0 + H_{\mathrm{ee}}$$

where H_0 models the electronic kinetic energy and interactions with the nucleus:

$$H_0 = \sum_{i=1}^{Z} \left[\frac{p_i^2}{2m} - \frac{Ze^2}{4\pi\epsilon_0}\delta(x_i) \right]$$

and H_{ee} models the electron-electron interactions:

$$H_{\mathrm{ee}} = \sum_{i<j} \frac{e^2}{4\pi\epsilon_0}\delta(x_i - x_j).$$

Make the ansatz that the wave function of the electrons is a product wave function:

$$\Psi(x_1, \ldots, x_Z) = \prod_{i=1}^{Z} \sqrt{\kappa}\mathrm{e}^{-\kappa|x_i|}.$$

Let κ be your lone variational parameter, for simplicity.

(a) As a warm up, do the case $Z = 1$.[1] Show that you recover the well-known energy *exactly* of the δ function well.

(b) Now do the case with general Z. Show that the variational bound on the ground state energy is given by

$$E_0 < -\frac{m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{Z(3Z+1)^2}{32}.$$

(c) How do you think we could improve upon our variational guess? You don't have to do any more calculations.

(d) Comment on whether or not this is a realistic model for an atom. What is the biggest problem with this model?

[1] Note that H_{ee} is irrelevant in this case.