quantum mechanics $\rightarrow$ approximation methods


## Quantum Bouncing Ball

In this problem we consider the quantum dynamics of a "bouncing ball," or a point mass of mass $m$ with quantum Hamiltonian

$$
H=\frac{p^{2}}{2 m}+m g z
$$

Furthermore, we are restricted to the half-space $z>0$ (i.e., the ball bounces off of the floor at $z=0$ ). Now, this problem is going to get a bit involved, and so right away we're going to want to work in dimensionless units.
(a) Explain how to scale position and energy from $z \rightarrow x$ and $E \rightarrow \epsilon$, respectively, so that the eigenvalue equation becomes

$$
\frac{\mathrm{d}^{2} \psi}{\mathrm{~d} x^{2}}=(x-\epsilon) \psi .
$$

(b) Use the WKB approximation estimate the eigenvalues $\epsilon$ for the above equation, with the appropriate boundary conditions.

This is the classical example of a problem which can be solved very accurately by the WKB approximation.
In fact, however, there are still some surprises to the quantum bouncing ball! To get further, we'll want to understand a bit more about the exact solutions. These are called Airy functions, and they are defined as the normalizable solution to the differential equation

$$
\frac{\mathrm{d}^{2}}{\mathrm{~d} x^{2}} \operatorname{Ai}(x)=x \operatorname{Ai}(x)
$$

It turns out one has the following integral representation:

$$
\operatorname{Ai}(x)=\frac{1}{\pi} \int_{0}^{\infty} \mathrm{d} t \cos \left(x t+\frac{t^{3}}{3}\right)
$$

The zeroes of the Airy function can be denoted by $-\beta_{n}$, where $\beta_{n}>0$ are discrete real numbers.
(c) What are the exact eigenvalues of the Hamiltonian? (You may still use dimensionless units.)

Our goal is now to look at the time-evolution of a "quasi-classical" state:

$$
\Psi(x, 0)=\left(\frac{1}{2 \pi \sigma^{2}}\right)^{1 / 4} \mathrm{e}^{-\left(x-x_{0}\right)^{2} / 4 \sigma^{2}}
$$

with $\sigma \ll x_{0}$. Define $n_{0} \equiv \min _{n}\left|\epsilon_{n}-z_{0}\right|$ and $T \equiv 2 \sqrt{x_{0}}$. Note that the classical trajectory of this particle would be ${ }^{1}$

$$
z_{\mathrm{cl}}(t)=\frac{2 x_{0}}{3}+\sum_{n=1}^{\infty}(-1)^{n} \frac{4 x_{0}}{\pi^{2} n^{2}} \cos \frac{2 \pi n t}{T}
$$

[^0]Finally, you may find the following list of identities useful:

$$
\begin{aligned}
\beta_{n} & \approx\left[\frac{3 \pi}{2}\left(n-\frac{1}{4}\right)\right]^{2 / 3}, \\
N_{n}^{2} & \equiv \int_{0}^{\infty} \mathrm{d} x \operatorname{Ai}(x)^{2} \approx \frac{\pi}{\sqrt{\beta_{n}}}, \\
0 & =\int_{0}^{\infty} \mathrm{d} x \operatorname{Ai}\left(x-\beta_{n}\right) \operatorname{Ai}\left(x-\beta_{m}\right) \quad \text { if } m \neq n, \\
\frac{2 N_{n} N_{m}(-1)^{n-m}}{\left(\beta_{n}-\beta_{m}\right)^{2}} & =\int_{0}^{\infty} \mathrm{d} x x \operatorname{Ai}\left(x-\beta_{m}\right) \operatorname{Ai}\left(x-\beta_{n}\right) \quad \text { if } m \neq n, \\
\frac{2 \beta_{n} N_{n}^{2}}{3} & \approx \int_{0}^{\infty} \mathrm{d} x x \operatorname{Ai}\left(x-\beta_{n}\right)^{2} .
\end{aligned}
$$

(d) Show that ${ }^{2}$

$$
\langle n \mid \Psi(t=0)\rangle=\frac{N_{n}}{\left(2 \pi \sigma^{2}\right)^{1 / 4}} \mathrm{e}^{-\left(\beta_{n}-x_{0}\right)^{2} / 4 \sigma^{2}}\left[1-\frac{x_{0}-\beta_{n}}{4 \sigma^{4}}+\frac{\left(x_{0}-\beta_{n}\right)^{3}}{24 \sigma^{6}}\right] .
$$

(e) Show that if $n_{0} \gg 1$ and $\left|n-n_{0}\right| \ll n_{0}$ :

$$
\beta_{n} \approx \beta_{n_{0}}+\frac{\pi}{\sqrt{\beta_{n_{0}}}}\left(n-n_{0}\right)-\frac{\pi^{2}}{4 \beta_{n_{0}}^{2}}\left(n-n_{0}\right)^{2} .
$$

(f) Ignore the quadratic term in the approximation of part (e). Show that then $\langle z(t)\rangle=z_{\mathrm{cl}}(t)$ by using your previous results and the Airy function identities.
(g) Now include the effects of the quadratic term and let $t=N T$. Show that if $N T^{3} / 4 \pi$ is close to an odd integer, then the quantum bouncing ball will be almost exactly out of phase with the bouncing oscillations of the classical ball.

[^1]
[^0]:    ${ }^{1}$ You do not need to find this explicitly. It is just an exercise in Fourier transforms.

[^1]:    ${ }^{2}$ Begin by using the integral representation of the Airy function. Perform the Gaussian integral over $x$. Then perform an approximate Gaussian integral over $t$, keeping only the lowest order term in $t^{3}$.

