Fake Power Laws

Scientists have a bad habit of *loving* to fit data to power laws. For example, given some data y and x, the idea is to set

$$y = ax^b$$

for some tunable a and b. However, sometimes these power laws are fake. Data that for an order of magnitude appears to fit a power law can actually be the consequence of a completely different function! In this problem, you will look at the functions

$$y(x) = \frac{1}{x + x_0} + 1$$

for $1 \gg x_0 > 0$. Does this function look like a power law for $x_0 < x < 1$?

To analyze this, we will try and find the "best" possible power law fit. Naïvely, we only get to tune a and b, but in practice we also get to tune the *scale* at which we look for the fit. So in fact, we get 3 parameters to tune. Now, let's see how much this $y(x)$ looks like a power law!

(a) Perform a Taylor expansion to all orders of $y(x)$ about the point $x = c > 0$.

(b) Perform a Taylor expansion to all orders of $Y(x) = ax^b$ about the point $x = c > 0$.

(c) Write down 3 equations for a, b and c, which correspond to the equality of the first 3 terms in the Taylor series for y and Y.

(d) Solve these equations numerically, and plot $y(x)$ and $Y(x)$ for a couple values of x_0 (e.g., $x_0 \sim 0.1$). What range is b in? How good is the fit? Do you think you could be confused by this if you were looking at experimental data?

(e) In what limit is the power law fit genuine?