Horse Race Gambling

At a horse race (the *very* old people version of sports), you place bets on various horses that are racing. If you bet money w_i on horse i, and horse i wins the race, then you will get $a_i w_i$ money back; otherwise, you get nothing. a_i is called the "odds" of horse i. We assume in this problem that $a_i > 1$.

Now, suppose we have a gambling addict betting on the horse race. He starts with money W_0, and will bet a fraction f_i of his money on horse i. Assume that horse i wins the race at time t with probability p_i, which is independent of t. He will also simply not bet a fraction f_0 of his money. If we define $Z_{i,t} = \mathbb{I}(\text{horse } i \text{ wins race } t)$,[1] then given that the addict has money W_{t-1} going in to race t:

$$W_t = f_0 W_{t-1} + \sum_{i=1}^{n} a_i f_i W_{t-1} Z_{i,t}.$$

We have assumed that f_i does not change between races, for simplicity. Now, by taking the log of both sides, we find that

$$\log W_t = \log W_0 + \sum_{s=1}^{t} \log\left(f_0 + a_i f_i Z_{i,s}\right).$$

The law of large numbers from probability theory tells us that the sum over the logs converges to its average value in probability as $t \to \infty$, implying that

$$W_t \to W_0 \alpha^t,$$

where

$$\alpha = \langle \log(f_0 + a_i f_i Z_{i,t}) \rangle.$$

If the gambling addict is smart, he will therefore try to maximize α. The optimal gambling strategy is constrained by the conditions that $f_0, f_i \geq 0$ and

$$\sum_{i=1}^{n} f_i + f_0 = 1.$$

(a) Using appropriate multipliers, write down the Kuhn-Tucker conditions for α.

(b) Discuss why the optimal choices of f_i are independent of time.

As you will show in this problem, the optimal strategy is greatly dependent on the value of the sum

$$\beta \equiv \sum_{i=1}^{n} \frac{1}{a_i}.$$

(c) Suppose that $\beta = 1$. Show that the optimal strategy is proportional gambling: $f_i \sim p_i$. Interestingly, this is independent of a_i.

[1] This is a random variable which is 1 if horse i wins race t, and 0 if horse i does not win race t. The $\mathbb{I}$ is called an indicator function.

(d) Suppose that $\beta < 1$. What is the optimal strategy?

(e) When $\beta < 1$, there exists a strategy with *zero risk* in the following sense: with probability 1, the gambler will grow his wealth: $W_t > W_{t-1}$. Find such a strategy. Is it optimal?

(f) Suppose that $\beta > 1$. Show that in this case, it is optimal to have $f_0 > 0$. While it is in general not easy to write down a closed form expression for f_i, describe the method one would use to find the optimal strategy. While you don't have to do this, it is certainly very easy to implement numerically.