Hyperbolic Geometry in the Disk

In this problem, we will explore hyperbolic geometry in disk $\mathbb{D}$. The **pseudo-hyperbolic distance** between two points $z_1, z_2 \in \mathbb{D}$ is defined as

$$D(z_1, z_2) = D(z_2, z_1) = \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|.$$

(a) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Show that $D(f(z_1), f(z_2)) \leq D(z_1, z_2)$.[1]

(b) Show that if $f \in \text{Aut}(\mathbb{D})$, then $D(f(z_1), f(z_2)) = D(z_1, z_2)$.

(c) Prove that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

This is called the **Schwarz-Pick Lemma**.

Now let's get a bit more exotic. We formally define a hyperbolic metric on $\mathbb{D}$, given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}z \mathrm{d}\overline{z}}{(1 - |z|^2)^2}.$$

Note that the right hand side of the above equation is a positive real number, so we can "take the square root" of both sides to find the infinitesimal length, $\mathrm{d}s$, between two points very close to each other in the disk. Now, let's define

$$\mathcal{C}(z_1, z_2) = \{\gamma(t) \text{ smooth}, \ \gamma(t) : [0, 1] \to \mathbb{D}, \ \gamma(0) = z_1, \ \gamma(1) = z_2\}.$$

(d) Given some trajectory $\gamma \in \mathcal{C}(z_1, z_2)$, explain why the distance between z_1 and z_2 along γ is given by[2]

$$d[\gamma] = \int_0^1 \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} \mathrm{d}t.$$

Because d_γ will usually depend on γ, we define the **hyperbolic distance** between z_1 and z_2 to be the smallest possible distance:

$$d(z_1, z_2) = \inf_{\gamma \in \mathcal{C}(z_1, z_2)} d_\gamma.$$

(e) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Prove that $d(f(z_1), f(z_2)) \leq d(z_1, z_2)$.

(f) Prove that, if $f \in \text{Aut}(\mathbb{D})$, for any pair of points, $d(f(z_1), f(z_2)) = d(z_1, z_2)$.

Now, let's find some expressions for hyperbolic distance!

(g) Let $s \in [0, 1)$. Show that

$$d(0, s) = \text{arctanh}(s) = \frac{1}{2} \log \frac{1 + s}{1 - s}.$$

(h) Generalize the above result to find $d(z_1, z_2)$ for any $z_1, z_2 \in \mathbb{D}$.[3]

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[1] Use automorphisms of $\mathbb{D}$ to perform a conformal map so that $f(0) \to 0$. Then apply Schwarz' Lemma.

[2] To do this, note that the definition of a distance would be the integral of $\mathrm{d}s$ along a curve.

[3] Use the result of part (g), along with appropriate automorphisms of the disk.