The Generator for Divisors

Let $d(n)$ be the number of divisors of $n \in \mathbb{N}$: i.e., the number of $k \in \mathbb{N}$ for which $\exists\ m \in \mathbb{N}$ such that $km = n$. Define

$$F(z) = \sum_{n=1}^{\infty} d(n) z^n$$

(a) Verify that the radius of convergence of this series is 1.

(b) Prove that

$$F(z) = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}.$$

(c) Let $0 < r < 1$, and $p, q \in \mathbb{N}$. Show there is some constant $c_{p,q}$ for which

$$\left| F\left(r \mathrm{e}^{2\pi \mathrm{i} p/q} \right) \right| \geq c_{p,q} \cdot \frac{1}{1-r} \log \frac{1}{1-r}.$$

(d) Can $F(z)$ be analytically continued past the unit disk $\{|z| < 1\}$?