The Partition Function of Number Theory

Number theorists define the partition function $p(n)$, for $n \in \mathbb{N}$, to be

$$p(n) = \text{number of distinct ways to write "}n = k_1 + \cdots + k_m\text{" for } k_i \in \mathbb{N}.$$

(a) Justify that a generating function $P(w)$ for the partition function is given by

$$P(w) \equiv \sum_{n=1}^{\infty} p(n) w^n = \prod_{m=1}^{\infty} \frac{1}{1-w^m}.$$

(b) Let $w = \mathrm{e}^{2\pi \mathrm{i}z}$, and consider the function $P(\mathrm{e}^{2\pi \mathrm{i}z}$. Verify this function is holomorphic for $\mathrm{Im}(z) > 0$?

(c) Find a contour C such that

$$p(m) = \int_C \mathrm{d}z\, \mathrm{e}^{-2\pi \mathrm{i} m z} \prod_{k=1}^{\infty} \frac{1}{1-\mathrm{e}^{2\pi \mathrm{i} k z}}.$$

(d) It turns out that $P(z)$ is a slight variant of a very famous function called the Dedekind η function:[1]

$$\eta(z) \equiv \mathrm{e}^{\mathrm{i}\pi z/12} \prod_{n=1}^{\infty} \left(1 - \mathrm{e}^{2\pi \mathrm{i} n z}\right).$$

In particular, one can show that this function satisfies

$$\eta\left(-\frac{1}{z}\right) = \left(\frac{z}{\mathrm{i}}\right)^{1/2} \eta(z).$$

Using this trick, show that for $z = x + \mathrm{i}y$ with y large that we can approximate

$$P\left(\mathrm{e}^{2\pi \mathrm{i}z}\right) \approx \left(\frac{z}{\mathrm{i}}\right)^{1/2} \mathrm{e}^{\mathrm{i}\pi(z^2+1)/12z}.$$

(e) Now, apply the method of steepest descent to the integral, in the limit that $n \gg 1$, and in the limit that y is large. To do this, begin by writing the integral as

$$p(n) \sim \int \mathrm{d}z \left(\frac{z}{\mathrm{i}}\right)^{1/2} \mathrm{e}^{-\mathrm{i}\kappa(z^2+1)/z}$$

and find the coefficient of proportionality in the integral, and the value of κ. Then, approximate the integral by finding the saddle, and justifying that the approximation of the previous part will hold at the saddle. For simplicity you do not need to pay much attention to worrying about the contour integral away from the saddle point. You should find that

$$p(n) \sim \frac{\mathrm{e}^{\pi\sqrt{2n/3}}}{n}.$$

Indeed, this is precisely the correct answer for the asymptotic form of $p(n)$ at large n up to small corrections, although it would take a lot more work to show that.

[1] This function is invariant under the modular group of the torus, and thus shows up in the theory of meromorphic functions on the torus. One prominent physical application of these functions is in string theory, where the torus corresponds to a closed string creating/annihilating itself.