Decimal Shift Map

We define a real number "(mod 1)" as

$$x \pmod 1 \equiv x - \lfloor x \rfloor$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Thus, $x \pmod 1 \in [0, 1)$. We define the decimal shift map as

$$f(x) = 10x \pmod 1.$$

This is a rare example of a chaotic system which can exactly be analyzed by hand! In some ways this chaotic map is overly easy to analyze. Nonetheless, we can ask the following interesting questions about the fixed points, stability, etc. of this mapping:

(a) How many fixed points of f are there?

(b) Show that there exists a periodic orbit of period p for each integer $p > 1$.

(c) Comment on the stability of each of these fixed points or periodic orbits.

(d) Show that if I pick x randomly from the uniform distribution on $[0, 1)$, the sequence $\{x_0, x_1, \ldots\}$ will be aperiodic with probability 1.

(e) One of the things we'll be interested in as we explore chaotic systems is the time scale for two trajectories which are close to diverge. Suppose that $|x_0 - x_0'| < \epsilon$. Find the time scale $\tau(\epsilon)$ for these two trajectories to diverge.