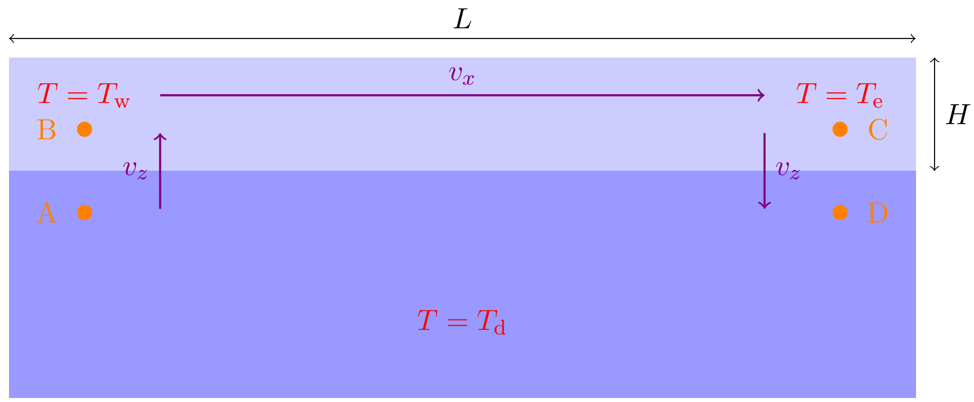
El Niño

In this problem, we will consider an extremely simple model for the formation of El Niño storms in the Pacific Ocean, as a consequence of a chaotic interplay between ocean temperatures and currents.



Let us consider the following toy model for the Pacific Ocean drawn above. The Pacific Ocean can be approximated as having two layers: a cold deep ocean layer at temperature T_{d}, which begins at a rough height H below the surface of the ocean, and a warmer upper layer of thickness H with variable temperature. The length of the ocean is L, and we make a temperature measurement of this upper layer: T_{w} at the west end (e.g., Japan), and T_{e} at the east end (e.g., California). When $T_{\mathrm{e}} - T_{\mathrm{w}} > 0$, fluid gets carried along the upper layer of the ocean from west to east, as shown in the diagram above. The nonlinear interplay between the fluid flow, which is described by vector field $\mathbf{v}$, and the temperatures will lead to chaotic dynamics, as we argue in this problem.

We begin with a heuristic discussion of transport effects in the ocean.

(a) We can approximate the flow of water in the ocean as incompressible, which implies that $\nabla \cdot \mathbf{v} = 0$. Use continuity to determine the relative scaling between v_x and v_z.

(b) Now, connect points ABC, and use a "finite difference approximation" to the transport equation of temperature: $\partial_t T + \mathbf{v} \cdot \nabla T = 0$, to argue that the convective rate of change of temperature is given by

$$\dot{T}_{\mathrm{w}}|_{\text{convection}} = \frac{v_x}{L}(T_{\mathrm{d}} - T_{\mathrm{e}}).$$

(c) Use a similar argument to determine $\dot{T}_{\mathrm{e}}|_{\text{convection}}$.

We also want to account for diffusive thermal effects. We can crudely do this by simply adding a linear decay term. However, since the surface of the ocean is affected by both the atmosphere and the deep ocean, it may not relax to the deep water temperature T_{d}, but a different temperature T_0 instead. We thus have

$$\dot{T}_{\mathrm{e}}|_{\text{diffusive}} = -\alpha(T_{\mathrm{e}} - T_0),$$
$$\dot{T}_{\mathrm{w}}|_{\text{diffusive}} = -\alpha(T_{\mathrm{w}} - T_0).$$

Finally, we approximate that the dynamics of the scale of east-west ocean currents is given by

$$\dot{v}_x = -\beta v_x + \frac{\gamma}{L}(T_\mathrm{e} - T_\mathrm{w}).$$

for positive constants β and γ.

(d) Since temperatures only enter these equations in difference relations, let us choose to set $T_\mathrm{d} = 0$. Define

$$\sigma = \frac{T_\mathrm{e} + T_\mathrm{w}}{2},$$
$$\delta = \frac{T_\mathrm{e} - T_\mathrm{w}}{2}.$$

Show that further rescalings of parameters and time, as well as combining the diffusive and convective contributions to the changes in temperatures, lead to the equations

$$\dot{v}_x = a\delta - bv_x,$$
$$\dot{\delta} = v_x\sigma - \delta,$$
$$\dot{\sigma} = 1 - v_x\delta - \sigma.$$

Determine the dimensionless parameters a and b in terms of L, γ, α, β and T_0, and discuss their physical meaning.

(e) For the Pacific Ocean, where El Niño storms occur, we have $L \approx 8000$ km. Crude estimates for the remaining parameters are $\gamma \approx 2\ \mathrm{m}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{K}^{-1}$, $\alpha \approx 7 \times 10^{-8}\ \mathrm{s}^{-1}$, $\beta \approx 2 \times 10^{-7}\ \mathrm{s}^{-1}$, and $T_0 - T_\mathrm{d} \approx 8$ K. Determine the relevant values of a and b for our model.

(f) Show that only when $a > b$ do fixed points exist. Find these fixed points, when they exist.

(g) Show that the fixed points of part (f) become unstable when

$$a \geq \frac{b^3 + 4b}{b - 2}$$

(h) Perform numerical simulations of these equations, using the values of a and b you found in part (e). Your simulations should suggest that this is a chaotic model. This chaotic dynamics in temperature corresponds to our simple model of the formation of an El Niño storm!

(i) El Niño storms appear in the Pacific Ocean, but not in the Atlantic Ocean, which has a typical length of about 4000 km. It is not clear what other parameters should vary significantly between these two oceans. With this fact in mind, discuss the feasibility of this model.