Fractal Basin Boundaries

Sometimes we deal with chaotic systems with a simple set of stable fixed points (whether it is finite, or just a smooth submanifold of phase space). In this case, the chaos comes in due to the sensitive dependence of which fixed point you end up at on initial conditions.

Let us consider an unspecified n-dimensional dynamical system with 2 fixed points, which we label A and B. Let $\Sigma \subset \mathbb{R}^n$ be the set of all points which are unstable fixed points of the dynamics, and for which on one side of Σ, the dynamics has $t \to \infty$ fixed point A, and on the other it has fixed point B.

(a) Explain why $\dim \Sigma \geq n - 1$.

(b) Suppose that we pick some point uniformly in $\mathrm{B}_\epsilon(x)$ (the ball of radius ϵ, centered at $x \in \mathbb{R}^n$) as our initial condition. Denote ρ to be the probability that a given ball contains points which will tend to both fixed points. Show that $\rho \sim \epsilon^{n-\dim \Sigma}$, where $\dim \Sigma$ is the box dimension.

(c) As a concrete example of this procedure, let us denote the set A to be the set of all points which would end up in A, and B for B. Consider choosing an initial condition in $(0, 1)$, and suppose that for some $0 < \lambda < 1$:

$$A = \bigcup_{m=0}^{\infty} \left(\lambda^{2m+1}, \lambda^{2m}\right),$$
$$B = \bigcup_{m=1}^{\infty} \left(\lambda^{2m}, \lambda^{2m-1}\right).$$

Show that, for this example,

$$\rho \sim \epsilon \log \frac{1}{\epsilon}.$$

This suggests that, crudely speaking, the box dimension of the fractal boundary set here is 0^+.

In fact, it is often the case that to numerically estimate the dimension of Σ, you would simply numerically perform the procedure outlined in part (b).