The Kepler Map

Let us consider the behavior of the electron in the hydrogen atom under the action of a very strong external field. For simplicity, we will treat the problem as classical, although it was primarily explored as a problem in quantum chaos. The Hamiltonian of a simplified version of the system can be written as

$$H = \frac{p^2}{2} - \frac{1}{z} + F\cos(\omega t),$$

where $z \in \mathbb{R}^+$ (we assume there is an infinite wall boundary at $z = 0$).

What is the resulting equation of motion? It's not hard to see that the electron will basically fall to the wall at $z = 0$, get reflected off, and then fall back in, etc. What makes this tricky is the fact that the oscillating force may be in different phases during each cycle. Let us denote by T_n the time between collisions n and $n+1$. For this problem, you should assume that $\omega T_n \gg 1$.

We will not study the exact solution of the equation of motion, but rather a discrete mapping of the energy E_n and a phase angle θ_n. The θ_n will simply track the phase of the oscillating force. Given a time step T, we then have

$$\theta_{n+1} - \theta_n = \omega T_n.$$

(a) We first begin by determining the discrete map governing this system, from the equations of motion. Show that

$$E_{n+1} - E_n = -\omega F \int_0^{T_n} \mathrm{d}t\; z(t)\sin(\theta_n + \omega t) = -2\omega F \sin\theta_n \int_0^{T_n} \mathrm{d}t\; z(t)\cos(\omega t).$$

(b) Find the proper relation between T_n and E_n by studying the equations of motion. Explain why you can neglect the oscillatory term in H if $\omega T \gg 1$.

(c) Why can you approximate that

$$z(t) \sim t^{2/3}$$

for $0 \le t \le T_n/2$. What about $T_n/2 \le t \le T_n$? Note that you will want to keep around the coefficients of proportionality.

(d) Conclude that the mapping is

$$E_{n+1} - E_n \approx \frac{2.58F}{\omega^{2/3}}\sin\theta_n,$$
$$\theta_{n+1} - \theta_n = \frac{2.22\omega}{|E_n|^{3/2}}.$$

You may find the values of the numerical constants above by evaluating things numerically. This is often called the **Kepler map**.

(e) Now explore this model numerically. By tuning F, and ω, can you find the onset of chaos?

(f) Can you find a switch between two different kinds of chaotic behavior in this map?