4D Chern-Simons Electromagnetism

In this problem, we will consider a Lorentz-violating theory of electromagnetism, with Lagrangian given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}q^{\mu}A^{\nu}F^{\rho\sigma}.$$

Treat q^{μ} as a constant vector in spacetime. Since q^{μ} picks out a preferred direction in spacetime, we break the manifest Lorentz invariance of our theory. The latter term is called a Chern-Simons term, and in 3D theories, the term $\epsilon_{\mu\nu\rho}A^{\mu}F^{\nu\rho}$ is an allowed term in a Lorentz-invariant Lagrangian which has applications in condensed matter theories.

(a) Show that the action is gauge invariant under $A_{\mu} \to A_{\mu} + \partial_{\mu}\lambda$ if all fields vanish at infinity.

(b) Find the equations of motion in terms of the physical fields $\mathbf{E}$ and $\mathbf{B}$, and comment on the results. What does the Chern-Simons term look like?

(c) Now, consider a propagating plane wave. Show that there are two modes of propagation with dispersion relation given by

$$k^{\alpha}k_{\alpha}k^{\beta}k_{\beta} = k^{\alpha}k^{\beta}q_{\alpha}q_{\beta} - k^{\alpha}k_{\alpha}p^{\beta}p_{\beta}.$$