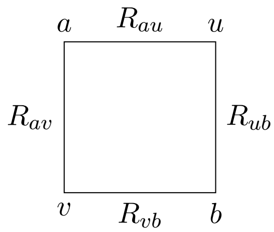
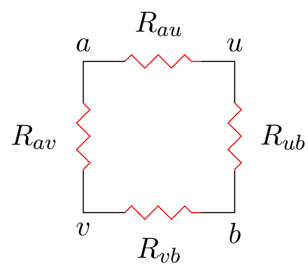
Thomson's Principle

For many complicated systems of real interest, it is extremely hard to calculate quantities of interest exactly, and we must instead find ways to approximate them. In this problem, we will prove and put to use a very powerful technique for estimating the effective resistance of a very large circuit, known as Thomson's principle.

Before discussing this technique, let us describe an arbitrary circuit of resistors in terms of a mathematical object called a graph. A graph consists of a set of points (nodes) $\{u_1, \ldots, u_n\}$ and a set of edges which connect two points: e.g., the edge (u_1u_2) connects the points u_1 and u_2. For our purposes, we will set $(u_2u_1) = (u_1u_2)$. A graph is a lot like a circuit: each node in the circuit where 2 different wires connect is a node in the graph, and each edge between nodes is a resistor. Of course, that does not completely specify things: to describe the circuit, we must also provide a function R_{uv} for every edge in the graph which describes the resistance.





Given a graph with resistances as above, and the nodes a and b, we define a flow from a and b to be any function of two nodes J_{uv} with the properties that $J_{uv} = -J_{vu}$ and $J_{wx} = 0$ if (wx) is not an edge of the graph. Furthermore, we demand that

$$\sum_{v\sim u} J_{uv} = \begin{cases} 1 & u = a \\ -1 & u = b \\ 0 & \text{otherwise} \end{cases} .$$

In the sum above, the notation $v \sim u$ means all nodes in the graph with (uv) an existing edge (alternatively, all nodes such that u and v are connected by a resistor). We see immediately that the current, I_{uv}, flowing down an edge, is precisely an example of a flow, in the case that we have connected a voltage source between a and b.

Thomson's principle states that the power function

$$P_{\mathrm{T}}[J_{uv}] \equiv \sum_{\text{edges}} J_{uv}^2 R_{uv},$$

when restricted to flows J_{uv} as defined above, has a unique minimum, that the unique minimum corresponds to the physical flow between a and b of unit current, and that this minimum of P_{T} is precisely the effective resistance R_{eff} between a and b. We begin by proving this theorem.

(a) Begin by showing that $P_{\mathrm{T}} = R_{\text{eff}}$ if the flow function $J_{uv} = I_{uv}$, where I is the true, physical current which would pass through the system, subject to the constraints of the theorem.

(b) Consider any set of edges $(u_1u_2), (u_2u_3), \ldots, (u_nu_1)$ which forms a loop.[1] Define the function

$$K_{xy} \equiv \begin{cases} 1 & xy = u_j u_{j+1} \\ -1 & xy = u_j u_{j-1} \\ 0 & \text{otherwise} \end{cases} .$$

Show that $I + \epsilon K$ is a flow function for any ϵ, and that $P_{\mathrm{T}}[I + \epsilon K] > R_{\text{eff}}$ if $\epsilon \neq 0$. Explain why this completes the proof.

(c) Having seen and given the mathematical proof of the theorem, justify it on physical grounds.

Now, we use Thomson's principle as follows. We guess a reasonable function J_{uv} obeying the proper constraints. Then, we try and find a reasonable bound for R_{eff} by computing $P_{\mathrm{T}}[J_{uv}] \geq R_{\text{eff}}$. However, Thomson's principle can only find us upper bounds to R_{eff}. A slight modification will allow us to obtain lower bounds as well. Let us define a cutset between a and b to be a subset of the edges (resistors) such that any path between a and b must pass through a resistor of the cutset.

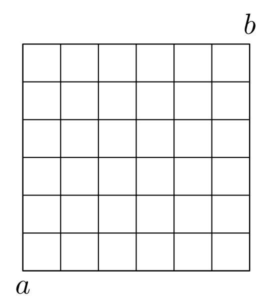
(d) Let A be a cutset of the edges. Explain why

$$\sum_{uv \in A} |I_{uv}| \geq 1.$$

(e) Let $A_1, \ldots, A_k$ be disjoint cutlets (no 2 cutlets share an edge). Use the Cauchy-Schwarz inequality to show that

$$R_{\text{eff}} \geq \sum_{i=1}^{k} \left[\sum_{uv \in A_i} \frac{1}{R_{uv}} \right]^{-1}$$

We're now ready to look at a hard circuit. Consider a 2 dimensional square grid of resistors, with N resistors with in each dimension. Each resistor has resistance $R_{uv} = r_0$. The question is what is R_{eff} between opposite corners of the square:



This problem is extremely hard to exactly solve, but we will determine how R_{eff} will scale with N when N is large using Thomson's principle.

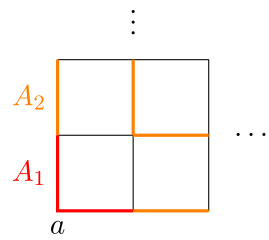
(f) Let's begin with the upper bound. A guess for the flow, inspired by an analogy to probability theory, is the following: the flow is always positive in the upwards or rightwards direction, and the sum of all incoming flows into point v is equal to $(n+1)^{-1}$, where n is the minimum number of steps that one must take to get from v to either a or b, whichever is smaller. Verify that this is a valid J, and show that

$$R_{\text{eff}} \leq c_1 r_0 \log N.$$

and estimate the constant c_1. Note that you should not need to precisely determine the flow along every edge: use approximations to simplify the calculation!

[1] Mathematicians call this a cycle.

(g) Now we turn to the lower bound. Using cutsets $A_1, \ldots, A_n$ as described in the picture below:



show that

$$R_{\text{eff}} \geq c_2 r_0 \log N$$

and estimate c_2.

Thus, without finding an exact answer, we have shown that $R_{\text{eff}} \sim r_0 \log N$ for the square. Note that R_{eff} scales very slowly with N, due to the large number of available paths for the current to take. In addition, the reason *why* it scales as $\log N$ should be clear from the physical intuition gained from the flows/cutsets we used to bound R_{eff}.

(h) As a final challenge with Thomson's principle, show that R_{eff} for the d dimensional hypercube with N resistors on each side (the $d = 2$ version was the square above) is bounded by a constant, independent of N, as long as $d \geq 3$.