1d Allen-Cahn Dynamics

The (Ising) Allen-Cahn equation in 1 spatial dimension is the following nonlinear PDE:

$$\alpha \partial_t u = \kappa \partial_x^2 u + ru - \lambda u^3.$$

It is used to model the dynamics of a magnet in its ferromagnetic state, or as a crude model for the dynamics of the liquid-vapor phase transition[1]. We assume that $\alpha, \kappa, r, \lambda > 0$.

(a) Show that there are no free parameters in the Allen-Cahn equation – all of them can be scaled away by redefining t, x and ϕ. The onset of the ferromagnetic phase transition corresponds to the limit $r \to 0$ – discuss what happens to the time and length scales in this limit.

(b) Show that there are three steady states, two of which are stable and one of which is unstable. Physically, the two stable states correspond to a pair of equivalent ferromagnetic ground states which break a parity symmetry, which will compete with each other. The ultimate ground state of this system will correspond to the system existing in either one of these steady states.

(c) Heuristically (but carefully) describe the short-time dynamics of this PDE. At $t \sim 1$, what do you expect $u(x,t)$ to look like, given arbitrary initial conditions?

Let's put your prediction in part (c) to the test by numerically simulating this PDE with random initial conditions on a periodic grid of size L. I would recommend using spectral methods. In particular, you should try and simulate the problem with random initial conditions:

$$\mathbb{E}[u(x,0)] = 0, \quad \mathbb{E}[u(x,0)u(y,0)] = A^2 \delta(x-y),$$

with $u(x,0)$ a Gaussian random variable with first two moments above. Now, of course you can't exactly do this numerically, but you can get close: I would start by choosing the initial conditions at each grid point to be i.i.d. random variables, and then by hand removing some fraction of the highest Fourier modes.

I won't tell you what the short-time dynamics is – however, the long-time dynamics turns out to be very tricky, so let me just say what should happen. By various arguments, physicists say that the number of ferromagnetic domains (for this problem, that's roughly the number of sign flips of u), should decrease as a power law as $t \to \infty$. Physically, this is the process by which the system chooses which of the two states it will end up in.

(d) Write code to solve this PDE, with the specified boundary conditions, using whatever method you would like. Present this code. Make sure that your code is stable, and present some plots of $u(x,t)$.

(e) Does your code's output agree with what you predicted in part (c), at short times?

(f) If your code is working, you should indeed find that the number of domains scales as $t^{-\nu}$, at late times, for some exponent $\nu > 0$. Present numerical evidence which is reasonably convincing, for what the value of ν is, and check that it is independent of both L and A. The correct answer is a very simple rational number, so please guess which simple fraction it is.[2]

[1] There is a slight subtlety here due to local conservation laws.

[2] There are theoretical arguments based on other ways of modelling ferromagnetism which predict the value of ν correctly.