Fireflies, Jetlag and Josephson Junctions

In this problem, we consider dynamics of a variable taking values on the circle (the manifold S^1). Let us denote its coordinate by θ, with $0 \leq \theta < 2\pi$. (By the usual rules, we simply take θ to be equivalent to $\theta + 2\pi n$, for any integer n).

The following ODE for θ,

$$\dot{\theta} = \alpha - \sin\theta$$

has been used to model a huge variety of natural oscillators, from Josephson junctions of superconductors, to the biological "locking" oscillators in fireflies and even in humans (which, when put out of sync by frequent travelers, can cause permanent jetlag!). Let's begin with the usual analysis. Without loss of generality, take $\alpha \geq 0$.

(a) Discuss the existence of fixed points for this flow. Explain why the dynamics of this system change greatly depending on whether $\alpha > 1$ or $\alpha < 1$.

(b) Sketch $\theta(t)$ for $\alpha > 1$.

(c) Sketch $\theta(t)$ for $\alpha < 1$.

Now, let us take $\alpha = 1 + \epsilon$, where $0 < \epsilon \ll 1$.

(d) Sketch $\theta(t)$, and very briefly comment.

(e) Determine why, to very good approximation, the period of this oscillator is

$$T = \frac{\pi}{\sqrt{2\epsilon}}.$$