Fish and Plankton

Let F be the the number of fish in a region of the ocean, and P be the number of plankton. A simple model of the population dynamics of these species is

$$\dot{F} = \alpha F \left[\frac{\beta P}{C + P} - F \right],$$
$$\dot{P} = \beta P \left[K - \frac{\zeta F}{C + P} - P \right].$$

(a) Describe the biological meanings of the various parameters.

(b) Describe how to nondimensionalize the dynamics, putting them in the form

$$\dot{F} = rF \left[\frac{P}{1 + P} - aF \right]$$
$$\dot{P} = P \left[k - P - \frac{F}{1 + P} \right].$$

(c) What is the meaning of the parameters r, a, and k?

Let's assume that (F_0, P_0) is a fixed point with $F_0, P_0 > 0$.

(d) Show that if $k < 1$ this fixed point is stable.

(e) Show that if $k > 1$ this fixed point may be unstable or stable. Describe how the value of r determines the stability.

Let's look at the case $k > 1$ in a bit more detail; we're interested in the case of instability.

(f) Determine the boundary of the region in the (a, k) plane where instability is possible. Sketch this region.

(g) Show that instability only occurs for $a < a_c$, where

$$a_c = \frac{4}{27}.$$