Heat-Salt Oscillator

In this problem, we will consider a simple model for how an instability could cause thermal oscillations in a body of water such as a lake. Let $T(t)$ denote the temperature of the body of water, and $S(t)$ the salinity of the body of water. Suppose that there are two sources with which this lake can exchange heat and salt. The first is an ocean, with $T = S = 0$, with which both heat and salt can be exchanged with the same rate. The second is an atmosphere, with $T = S = 1$, and while heat can be exchanged quickly, salt is exchanged very slowly. We postulate the following set of dynamical equations for this system:

$$\dot{T} = 1 - T - K(\gamma S - T)T$$
$$\dot{S} = \beta(1 - S) - K(\gamma S - T)S$$

with $\beta \ll 1$, and

$$K(x) = \begin{cases} cx & |x| < 1/c \\ \mathrm{sign}(x) & |x| > 1/c \end{cases} .$$

The ansatz for K goes as follows: only a relative difference between S and T can lead to exchange of heat and temperature, because fluid tends both towards the cooler body and the more salty body, and these effects can "cancel". This is of course, not a perfect assumption, but it is a moderate starting point.

(a) Classify the possible fixed points of this dynamical system, depending on the values of the parameters.

(b) Suppose there is 1 fixed point. Show that this model is then a relaxation oscillator. What parameter choices would make this possible, and what are their physical consequences?

(c) Sketch the phase plane flows if this model is a relaxation oscillator.

(d) You may be a bit concerned by the idea that temperature could oscillate. Is this unphysical? Comment on the feasibility of such a scenario.