Inhibition of Bacterial Respiration

Suppose we have a colony of bacteria in a petri dish, where the concentration of oxygen is given by x, and the concentration of nutrients is given by y. Of course, to make physical sense, we require $x, y > 0$. A theoretical proposal a while ago was that when the concentration of oxygen becomes *too large*, the cell actually began to consume less nutrients. This model will suggest that such behavior will have experimentally observable consequences.

To start off, let's consider the dynamics of respiration without inhibition. Suppose we add nutrients at some rate $A > 0$, and oxygen at some rate $A + B$. A crude model for the non-dimensionalized dynamics is that

$$\dot{x} = A + B - x - xy,$$
$$\dot{y} = A - xy.$$

This model basically says that the oxygen dissipates from the water on its own, and that both oxygen and nutrients are consumed by the bacteria proportional to their concentration. These are pretty simplistic assumptions and are the simplest reasonable ones we could make.[1]

(a) Explain why, for the model to make physical sense, $B > 0$.

(b) Show that there is a unique fixed point (x^*, y^*), and find it in terms of A and B.

(c) Show that this fixed point is stable: thus, the dynamics simply tends to an equilibrium.

Thus, nothing interesting happens in this model.

However, that is not what is experimentally observed. Experimentally, the concentrations x and y actually *oscillate*. A model which was proposed to give such oscillations was that the rate of nutrient consumption is suppressed at large x, yet mostly unaffected at small x. We can describe this by

$$\dot{x} = A + B - x - \frac{k^2 xy}{k^2 + x^2},$$
$$\dot{y} = A - \frac{k^2 xy}{k^2 + x^2}.$$

(d) Explain why, for the model to make physical sense, $B > 0$.

(e) What is the scale of x on which the nonlinear inhibition of respiration becomes large?

(f) Show that there is a unique fixed point of the dynamics.

(g) Sketch the nullclines and show the flows in the phase plane.

(h) Find values of the parameters such that the dynamics tends towards a limit cycle, and is a nonlinear oscillator. Remember to prove that the limit cycle does in fact exist!

[1] Since the bacteria require oxygen to generate ATP, the energy source which powers the cell to break down the nutrient molecules, we cannot have nutrients being consumed without oxygen.

This model therefore predicts that the rate of respiration in a bacterial colony can oscillate due to oxygen inhibiting the process when it becomes too highly concentrated. While no pretense was made of actually mapping this model to a realistic model of the dynamics of respiration, the basic idea is that nonlinear inhibition *could* be a cause for such a slowdown, and some nontrivial process is required to generate oscillations.