Non-Uniqueness of Solutions

Not all first order ODEs have a unique solution. Consider the ODE

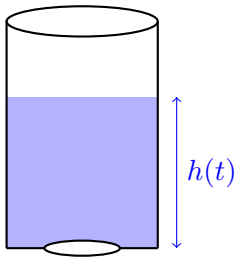
$$\dot{x} = |x|^{p/q}$$

with $p, q \in \mathbb{N}$ and relatively prime. Suppose the initial condition is $x(0) = 0$.

(a) Show that if $p < q$, then there is an infinite family of solutions, characterized by a solution "taking off" from 0 at a finite time t_0.

(b) Show that if $p > q$, then the solution $x(t) = 0$ is unique.

Sometimes the non-uniqueness of physical conditions has a physical meaning. Consider the following problem: a cylindrical bucket of cross sectional area A has a small hole of area a at the bottom. It is filled with water of height $h(t)$ which leaks out of the bucket under the effect of Earth's gravitational field, g:



(c) Show that the height of the water in the cup obeys the ODE

$$\dot{h} = -\frac{a}{A}\sqrt{2gh}.$$

(d) Comment on the physical meaning behind the non-uniqueness of the solution $h(0) = 0$ (considering running the equation backwards in time...).