Overfishing

Suppose we have fish in the ocean, whose population is proportional to x (a scaled variable); thus, we only are interested in dynamics for $x \geq 0$. We assume that naturally, the fish have logistic growth. We also assume that humans are catching the fish for food. The ODE for the growth rate of the population is

$$\dot{x} = x(1-x) - \frac{ax}{x+b}.$$

(a) We assume $a, b > 0$. Describe the "physical" meaning of the variables a and b.

(b) Explore the dynamics of the model for various a and b. Consider all possibilities. For each a and b, thinking about the fixed points, their locations, and their stability. Then, draw the (a, b) plane, and based on the behavior of the dynamics, label regions with either "extinction", "danger", or "sustainable".

Now, let's use this model to make some recommendations for the humans.

(c) Ignoring that "sustainable" sounds nicer than "danger", why should humans stay in this region of the (a, b) plane? Think practically, not mathematically, here.

(d) How should society choose a and b to maximize the fish they can catch per year while remaining in the "sustainable" region?

(e) Briefly comment on any lessons learned from the model – is damage irreversible? Once again be practical, not mathematical.