Population Growth with Aging

Let us consider a population described by $n(t,a)$, where n is the population density in age a at time t. If we measure a and t in the same units of time, then we have

$$\frac{\partial n(t,a)}{\partial t} + \frac{\partial n(t,a)}{\partial a} = -r(a)n(t,a),$$

where $r(a)$ is an age dependent death rate. The boundary conditions for this PDE, which is defined on $\mathbb{R}^+ \times \mathbb{R}^+$, are given by $n(0,a) = n_0(a)$ and

$$n(t,0) = \int_0^\infty \mathrm{d}a\, b(a)n(t,a).$$

Thus, the rate of birth of new individuals depends on the age distribution at time t.

(a) Show that the solution $n(t,a)$ has a solution given by, for a function $R(a)$ which you are to determine,

$$n(t,a) = \begin{cases} n_0(0, a-t)\mathrm{e}^{R(a-t)-R(a)} & a > t \\ \int_0^\infty \mathrm{d}a'\, b(a')n(t-a, a')\mathrm{e}^{-R(a)} & a < t \end{cases}.$$

Unfortunately, since this is a recursive relation, it does not count as an exact solution.

(b) In general, we do not necessarily care about the precise $n(t,a)$ – rather, we are interested in the dominant $t \to \infty$ behavior. Let us assume, for now, that the solution is of the form

$$n(t,a) \to \mathrm{e}^{\gamma t} n_\infty(a) \qquad (t \to \infty).$$

Show that

$$n_\infty(a) = n_\infty(0) = \mathrm{e}^{-R(a)-\gamma a}.$$

(c) Use the boundary condition at $a = 0$ to relate γ to $R(a)$ and $b(a)$.

(d) Show that the simple case of

$$\begin{aligned} r(a) &= \rho \\ b(a) &= \frac{\beta}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(a-\mu)^2}{2\sigma^2}\right] \end{aligned}$$

has an exact expression for γ, in the limit of $\mu \gg \sigma > 0$. Find it and comment on any interesting features, in terms of the biology.

(e) Write and show some numerical code to find a numerical solution to this PDE, in whatever language you'd like.

(f) Using the numerics to simulate the case solved in part 4, discuss whether you think the similarity solution was a good approximation. In doing this, you should consider the effects (if any) of the initial condition.

(g) Let's try and simulate a more realistic choice of $r(a)$ (the b is probably a decent approximation). Consider the choice

$$r(a) = \lambda e^{a/\alpha}.$$

Discuss what you find numerically happens as you tune λ, β, α, μ and σ. Do you expect that only certain ratios of these quantities are relevant? Try and say a bit theoretically, but you don't need to do any serious calculations. Can you find any interesting phenomena?