Rock Paper Scissors

Let us consider an evolutionary game where players can exist in a finite number of states in some space Ω, where there are n states. We can describe the interactions of each player by a **payoff matrix** Π_{ij}, where entry Π_{ij} describes the payoff/utility that player i gets after an interaction with j. Note it is not the case that Π is a symmetric matrix, in general. We assume that in general, each player has 1 interaction per unit time with a randomly chosen player. Finally, we assume that the probability of being in state $i \in \Omega$ grows in time with a rate proportional to the expected payoff as a player in state i compared to the average; thus, if p_i denotes the probability that a randomly chosen player is in state i

$$\dot{p}_i = (\langle \Pi | i \rangle - \langle \Pi \rangle)\, p_i$$

Note that implicit in this construction is that the number of players is effectively infinite, so that fluctuations and stochastic effects are suppressed.

To make things a little more interesting, let us also allow for players to randomly "mutate" into other states: we assume that this happens μ times per unit time. During mutation, we assume that a player is equally likely to choose any new state to enter (including the one he came from, for simplicity).

(a) Show that the equations of motion are

$$\dot{p}_i = \mu(1 - np_i) + p_i \sum_j \Pi_{ij} p_j - p_i \sum_{j,k} \Pi_{jk} p_j p_k.$$

(b) Verify that $\sum_i p_i(t) = 1$.

Obviously for arbitrary μ and Π, finding the locations of the fixed points simply amounts to solving these polynomial equations simultaneously, a task which is not easy. However, there is one fixed point which we find particularly interesting: the one in which all $p_i^* = 1/n$.

(c) Describe the requisite property of the payoff matrix, Π, which leads to this being a fixed point.

(d) Explain why, for this class of Π, the choice of μ does not alter the location of this fixed point.

(e) It is also true that if the symmetric fixed point is not an exact fixed point, there may be a fixed point very close to the symmetric point. Argue that this is what happens for large enough μ.

(f) Roughly describe a bound on μ for arbitrary Π such that this near-symmetric fixed point will be stable.

(g) While your bound on μ may not be precise, comment on its behavior with respect to n.

While the general problem may be tricky to discuss exactly, you might think that by looking at the cases with small n, one can still observe many interesting phenomena.

(h) Explain why the dynamics in the case of $n = 2$ (i.e., a two state game) is uninteresting in that dynamics always relaxes to an equilibrium.

If the case $n = 2$ is uninteresting, let's try the case $n = 3$. For example, let's consider the following game based on rock paper scissors: we take the payoff matrix to have entries

$$\begin{aligned}
\Pi_{11} &= \Pi_{22} = \Pi_{33} = 0 \\
\Pi_{12} &= \Pi_{23} = \Pi_{31} = 1 \\
\Pi_{21} &= \Pi_{32} = \Pi_{13} = -\epsilon
\end{aligned}$$

where $\epsilon > 0$. ϵ thus corresponds roughly to the ratio between how much people don't like losing vs. enjoy winning. By setting 1 to be rock, 2 to be scissors and 3 to be paper, we can observe the general nature of the rock paper scissors game evident in this model.

(i) Why is the symmetric point with $p_1 = p_2 = p_3 = 1/3$ a fixed point?

(j) Let $a = p_1$ and $b = p_2$. Show that

$$\begin{aligned}
\dot{a} &= a\left(1 - a - (1+\epsilon)b - (1-\epsilon)\left(ab + (a+b) - (a+b)^2\right)\right) + \mu(1 - 3a), \\
\dot{b} &= b\left(\epsilon b - (1+\epsilon)(1-a) - (1-\epsilon)\left(ab + (a+b) - (a+b)^2\right)\right) + \mu(1 - 3b).
\end{aligned}$$

(k) Sketch in the (ϵ, μ) plane the regions of the plane which lead to different behaviors for the dynamics of the system (near this fixed point, at least!).

(l) What types of behavior must people exhibit for this system to have nontrivial dynamics as $t \to \infty$? Is there an intuitive explanation for your results?