1D Poisson-Boltzmann Equation

The Poisson-Boltzmann equation shows up in the study of electrostatics in salty solutions. In a 1D system, it is an ODE of the form

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{2nq}{\epsilon_0}\sinh\frac{q\varphi}{k_\mathrm{B}T}$$

where φ is the electric potential, n is the ion density, q is the ion charge, T is the temperature, and ϵ_0 and k_B are constants.

(a) Show that a proper choice of rescaling of both φ and x can remove all free parameters from the Poisson-Boltzmann equation. Describe how the characteristic length scale and potential scale of the system change with T, n and q.

(b) It turns out this equation has an exact solution on the line $[0, \infty)$. Let us choose our boundary conditions so that $\varphi(0) = \psi > 0$, and $\varphi(\infty) = 0$. Show that, in dimensionless units:

$$\varphi(x) = 2\mathrm{arctanh}\left[\tanh\left(\frac{\psi}{2}\right)\mathrm{e}^{-x}\right].$$

(c) Study these solutions asymptotically. What happens at large x? Show that the behavior at small x depends on the value of ψ. After you make your predictions, plot the solutions of part (b) numerically and compare to your predictions.