SIR Epidemic

The SIR epidemic is a simple model for a disease epidemic which has proved to be the basic building block of a huge part of quantitative epidemiology. It goes like this: let S be the number of susceptible individuals, I the number of infected individuals, and R the number of "removed" (dead) individuals. We assume that the rate of infection is proportional to the number of infected individuals, but once an individual has been infected, their death rate is independent of their surroundings. In equations, this reads

$$\begin{aligned}\dot{S} &= -\lambda I S,\\ \dot{I} &= \lambda I S - \mu I,\\ \dot{R} &= \mu I,\end{aligned}$$

with $\lambda, \mu > 0$ positive fixed constants.

(a) Show that $N = S + I + R$ is a constant of motion. Explain the intuitive interpretation of this fact.

(b) Plug in for S using the previous part. Then, find an expression for $I(R)$ by integrating $\mathrm{d}I/\mathrm{d}R$.

(c) Show that after rescaling I and t (if necessary), we can thus reduce the dynamics to a single first order ODE

$$\dot{I} = a - bI - \mathrm{e}^{-I}.$$

Give the expressions for a and b, and describe any reasonable bounds on these parameters.

(d) Discuss the flows of this nonlinear ODE, describing fixed points and their stability.

(e) Justify the statement that we will have a disease "epidemic" if the maximum of I occurs at a time $t > 0$. Show that the maxima of I occurs at the same time of the maximum of $\dot{R}$ and $-\dot{S}$.

(f) Qualitatively sketch H, S and D for $b \gg 1$, for $|a - 1| \ll 1$. Show that in this case, there is an epidemic.

(g) Qualitatively sketch H, S and D for $b \gg 1$, for $|a - 1| \ll 1$. Show there is no epidemic.