Spruce Budworm Outbreak

In Eastern Canada, there is a certain species of insect called the spruce budworm, which enjoys eating fir trees. In this problem, we will explore how an outbreak of this insect species can decimate an entire forest.

Let's begin by looking at the fast dynamics of the insect growth. Let $B(t)$ be the number of budworms at time t. The budworms grow exponentially, but only to a certain extent before they are capped by resources. In addition, we assume that there are birds which are predators, and eat the budworms. We assume that the bird population rapidly shifts its diet depending on the budworm population. We can approximate that B obeys an equation like

$$\dot{B} = RB\left(1 - \frac{B}{K}\right) - \frac{\alpha B^2}{\beta^2 + B^2}.$$

The first term represents population growth of the budworms; the second the reduction in population due to predatory birds.

(a) Explain the physical (in this problem: biological!) meaning of R, K, α and β.

(b) Explain how to scale t and B such that we are left with a scaled equation for B of the form

$$\dot{B} = rB\left(1 - \frac{B}{K}\right) - \frac{B^2}{1 + B^2}.$$

(c) Show that depending on the values of r and K, there can be to 2 to 4 fixed points.

(d) The ecologically favorable situation is to be in a scenario where there are 4 fixed points. If you can pick between r and K (separately) being either large or small compared to 1, which would you pick?

(e) Given the choice above, give estimates for the values of b at the 2 stable fixed points.

Now, suppose that for some reason the bird population slowly begins to decrease. This will cause r to slowly increase, but we assume that k is fixed.

(f) Without quantitative analysis, explain why there is a critical value of r, r_c, such that if $r > r_c$ the budworm population will drastically increase.

Suppose that, on a time scale very slow compared to the dynamics of b, r is increased beyond r_c, and then decreased to its initial value.

(g) Plot $r(t)$ and $b(t)$ side-by-side. (There is no correct answer for $r(t)$ other than it needs to increase and decrease in the above manner.)

(h) Comment on the ecological significance of the result.

We now model the dynamics of the forest. Let $F(t)$ be the number of fir trees and $Z(t)$ be the number of "resources" available in the forest. We assume that the number of resources available grows until it reaches a stable point on its own, but that the budworms decrease the resources available at a rate

proportional to the number of insects per tree. The number of fir trees grows until it reaches a stable point proportional to the number of resources:

$$\dot{F} = \zeta F \left(1 - \sigma \frac{F}{Z}\right),$$
$$\dot{Z} = \mu Z \left(1 - \frac{Z}{C}\right) - \nu \frac{B}{F}.$$

(i) Explain the meaning of C, μ, ν, σ and ζ.

(j) Explain how to scale F and Z so that

$$\dot{F} = \epsilon F \left(1 - \frac{F}{Z}\right),$$
$$\dot{Z} = \eta \left[\theta Z(1 - Z) - \frac{B}{F}\right]$$

in terms of new parameters ϵ, η, θ.

From here forth, assume that $1 \gg r \gg \eta \gg \epsilon$. Let us further assume that a drop in the bird population causes a budworm outbreak, as we did earlier.

(k) Explain why, depending on the various parameters of the problem, there are two fundamentally different behaviors for the flows in the F, Z plane. Sketch the nullclines of F and Z for each case and the direction of the flows, commenting on the stability of any fixed points.

Assume that the parameters are such that there are 2 stable fixed points, and that the forest is at the fixed point where F and Z are larger.

(l) There is a critical value of θ, θ_c, for which the result of the outbreak becomes drastically different for $\theta > \theta_c$. Find θ_c exactly, in terms of r and K.

(m) Suppose that $\theta < \theta_c$ (but not by much!). Sketch the dynamics in the F, Z plane just after the outbreak.

(n) Sketch the dynamics for $\theta > \theta_c$ (but not by much!).

The results of the outbreak for $\theta > \theta_c$ are catastrophic. This problem is meant to show you how in ecology very slight changes in populations of one species can wipe out entire other ones.