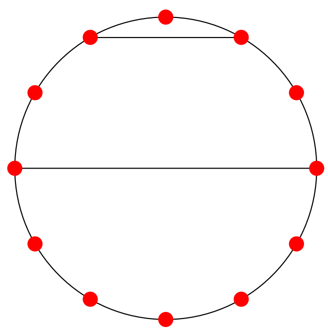
6 Degrees of Separation

It has become a quip that if two people are selected at random from Earth's population, they can be connected by a chain of usually no more than six friends, where each "link" is a fairly strong friendship. In fact, this phenomenon has significant evidence supporting it, and furthermore it can even be predicted theoretically. Let us consider a population of N individuals. Typically, friends are "clustered": i.e., most of my friend's friends are also my friends. Let's assume that, as shown below, individuals can be arranged on the edge of a ring, and are friends (i.e. connected) to their K nearest neighbors on each side. However, let's also place a small number of "exotic" edges at random between two nodes; since we take $K \ll N$, we neglect the possibility that we connect two nodes which were already friends. Let us denote by p the probability that any given node has such an exotic friend. We take $p \ll 1 \ll pN$. Below we see an example with $p = 0.25$, $K = 1$, and $N = 12$:



The network above is called a "small world" random graph. A graph is a mathematical structure where nodes are connected by edges: in the above picture, each node is a red dot, and each edge is a black line connecting a pair of nodes. In our social networking example, a node corresponds to a person, and an edge to a friendship. In this problem, we will describe such a small world graph at mean field level. Let us consider the subsets of individuals which are a length of t away from a given starting node, where t counts the number of edges required to get from one node to the other. Let's define

$$\mu(t) = 1 - \frac{\text{number of people connected after } t \text{ steps}}{N},$$

$$\nu(t) = \frac{\text{number of clusters after } t \text{ steps}}{N}.$$

By cluster, we mean a set of nodes along the outside of the ring, such that every node in between the end nodes of the cluster is within t steps of the original node. We will approximate that t is a continuous variable.

(a) Justify the following approximate equation:

$$\dot{\mu} = -2K\nu.$$

(b) There are two ways that the number of clusters could change. Firstly, two clusters could merge because the endpoints of the clusters come together. Secondly, a new cluster could form because of one of the “exotic” friendships leading to the formation of a new cluster at a new spot on the ring. Using this intuition, justify the following approximate equation:

$$\dot{\nu} = 2K\nu\left[p\mu - \frac{1}{\mu}\left(\nu - \frac{1}{N}\right)\right].$$

(c) The above equations are exactly solvable, at least to the extent we are interested in. Show that

$$\nu = \frac{1}{N} + p\mu(1-\mu).$$

(d) Plug in for $\nu(\mu)$ into $\dot{\mu}$, and find an expression for $t(\mu)$.

(e) What we really want to do is compute the average number of steps, L, it will take to get from one node to a randomly chosen node in the graph. Show that

$$L = \int_0^1 \mathrm{d}\mu\, t(\mu).$$

(f) Combine the previous two parts to show that

$$L = \frac{g(Np)}{Kp}$$

where

$$g(x) = \sqrt{\frac{x}{x+4}}\operatorname{arctanh}\sqrt{\frac{x}{x+4}}.$$

(g) Simplify the answer to the previous part in the limit when $Np \gg 1$.

(h) Now, let’s see if our answer is reasonable. Let’s suppose that $K \sim 100$, and $p \sim 0.01$: thus, everyone has about 1 exotic friend, and about 100 friends. There are about $N \approx 7 \times 10^9$ people in the world. With these numbers, what would be the value of L? Given your answer, comment on whether or not you think the idea of “6 degrees of separation” is reasonable.