Bacterial Soliton

A traveling group of bacteria is looking for nutrients. Let us denote by $B(\mathbf{x}, t)$ the density of bacteria at point $\mathbf{x}$ at time t; denote by $N(\mathbf{x}, t)$ the density of nutrients. We assume that the bacteria eat nutrients at a constant rate r:

$$\frac{\partial N}{\partial t} = -rB.$$

The change in bacterial density is given by the divergence of the flux of bacteria:

$$\frac{\partial B}{\partial t} = \nabla \cdot \left[k\nabla B - a\frac{B}{N}\nabla N\right].$$

There are two things which cause bacterial fluxes: the first term corresponds to the bacteria tending to cluster together (namely, if more bacteria are in a region, that implies more food). The second term corresponds to the fact that bacteria tend to move towards regions of higher nutrient density, and that the rate should increase as the bacteria per nutrient increases. k and a here are simply positive proportionality constants. We have assumed the simplest possible form for the PDEs consistent with these general ideas.

(a) Show that the parameters r and k can be scaled away, but a cannot.

From now on, set $r = k = 1$. Keep a general, but for simplicity you can assume $a \neq 1$, as this is a special case. Let us begin by assuming that there is one spatial dimension, and then look for a soliton solution: i.e., assume solutions are only functions of $z = x - ct$.

(b) Reduce the dynamics of the system to a pair of first order ODEs for $B(z)$ and $N(z)$. To do this, you will need to assume reasonable boundary conditions: assume that $B \to 0$ and $\mathrm{d}N/\mathrm{d}z \to 0$ as $|z| \to \infty$. Thus, the bacteria soliton is localized.

(c) Find exact expressions for $B(z)$ and $N(z)$, assuming we scale $N(\infty) = 1$.

(d) Conclude that

$$\frac{\mathrm{d}N}{\mathrm{d}z} = \frac{c}{a-1}\left(N - N^a\right).$$

Now that we have reduced the problem to a first order ODE, we can understand the qualitative behavior of the solution.

(e) Show that in the case of $a = 2$, the problem is exactly solvable, and (up to z translation)[1]

$$B(z) = \frac{c^2 \mathrm{e}^{-cz}}{(1+\mathrm{e}^{-cz})^2}$$
$$N(z) = \frac{1}{1+\mathrm{e}^{-cz}}.$$

(f) Show that for large z, $1 - N$ and B both fall off asymptotically as $\sim \mathrm{e}^{-\lambda z}$ and find an expression for λ. Consider all regimes of a.

[1] A similar technique would work for $a = 3, 4, \ldots$, but is a lot more work in these cases.

(g) Show that for large $-z$, if $a > 1$, B and N behave like $e^{\mu z}$, and find an expression for μ.

(h) What happens for $a < 1$?

(i) Sketch $B(z)$ and $N(z)$ when $a < 1$, as well as when $a > 1$.

(j) Suggest the biological mechanism for the radical departure in behaviors for $a < 1$ vs. $a > 1$.