Bottlenecks

Consider the ODE

$$\dot{x} = \epsilon + \eta x^{2n}$$

with $n > 0$ an integer, ϵ small, and $\eta > 0$ positive. This system is an example of a bottleneck in an ODE, where the system becomes "trapped" near $x = 0$, at a point which is almost, but not quite, a fixed point.

(a) Show that the "passage time" through this fixed point is

$$\tau = \frac{c_n}{\epsilon^{1-\frac{1}{2n}} \eta^{\frac{1}{2n}}}$$

with c_n dependent only on n.

(b) Show that

$$c_n = \frac{\pi}{2n \sin \frac{\pi}{2n}}$$

by performing a contour integral over a pie slice shaped contour.