Driven First Order Dynamics

Consider the first order inhomogeneous ODE

$$\dot{x} + \lambda x = f(t)$$

with $f(t)$ an arbitrary smooth function which is periodic with period T: i.e., $f(t) = f(t+T)$, and $\lambda \in \mathbb{R}$ a constant. Does a stable limit cycle of period T exist for this mapping: i.e., is there a solution $x^*(t)$ with $x^*(t) = x^*(t+T)$?

It is easiest to analyze this question by considering the map $P : \mathbb{R} \to \mathbb{R}$ with the property that if $x(0) = A$, then $x(T) = P(A)$. Of course, since the solution $x(t)$ to the ODE is unique, this is a well-defined map.

(a) Find an expression for the map $P(A)$.

(b) Show that there is a unique fixed point A^* where $P(A^*) = A^*$ and find an expression for it. Thus, prove that a limit cycle exists.

(c) Analyze the stability of the map P under small perturbations around the fixed point. In what cases is the limit cycle stable?