# Random Field Ising Model on Complete Graph

Many problems in computer science can be phrased as finding the ground state of an Ising "spin glass" model. This is a very useful analogy for many NP problems. However, not all computer science problems are hard. Does this mean that some Ising "spin glasses" are easy to solve? In this problem, we will explore this further.

Let's begin with a simple problem in computer science: finding the largest number out of a list $n_1, \ldots, n_N$ of numbers. This is a very simple algorithm which runs in $\mathrm{O}(N)$ time (we are assuming that the list is not ordered). Now, consider the quadratic energy functional

$$H = A\left(1 - \sum_{i=1}^{N} x_i\right)^2 - B\sum_{i=1}^{N} n_i x_i$$

where $x_i = 0, 1$ are binary variables associated to each number $n_i$ in the list.

(a) Show that if $A$ and $B$ are chosen properly, finding the ground state of this model is equivalent to solving the searching problem.

(b) Show that by an appropriate change of variables, this is equivalent to the random field Ising model on a complete graph.

The fact that the random Ising field model, for a special choice of magnetic fields, with uniform antiferromagnetic couplings on a complete graph is the same thing as solving a very simple computing problem suggests that some instances of the random field Ising model should be easy to solve. Now, let us consider the following random field Ising model on a complete graph:

$$H = J\sum_{i \neq j} s_i s_j - \sum_i h_i s_i.$$

(c) Rearrange the quadratic term so that the quadratic is a function only of the total magnetization.

(d) Suppose that $J \ll h$, typically. What is the requirement on $J$ such that the ground state of the random field Ising model can be found in $\mathrm{O}(N)$ time?

(e) Now, suppose that $J \gg h$. Show that in this case, again, the ground state of this Ising model can be found in $\mathrm{O}(N)$ time. What is the actual requirement on the relative sizes of $J$ and $h$?