The Kicked Rotator

The kicked rotator is a very simple, but interesting, model for a driven quantum mechanical system. Consider a quantum particle on S^1 with Hamiltonian

$$H(t) = \frac{L^2}{2} + K\cos\theta \sum_{n\in\mathbb{Z}} \delta(t-n)$$

with L^2 the angular momentum operator on the circle, and $\cos\theta$ a position operator.

Since $H(t)$ is time-dependent, the concept of energy eigenstates no longer makes sense. Indeed, the external driving force will add energy to the system, so instead what we will be interested in is how the kicking evolves the rotor in time.

(a) Let $F = U(1)$ be the time evolution operator through a period of time 1. Find an expression for F, assuming the starting time to be 0^+.

(b) What is special about the eigenvectors of F? Why are we interested in them?

(c) In the basis of the eigenvectors of L^2, show that

$$\langle n'|F|n\rangle = \sum_{n'} J_{n-n'} \mathrm{e}^{\mathrm{i}n^2/2}$$

and find an expression for $J_m(K)$.

(d) Suppose that $F|\alpha\rangle = \mathrm{e}^{\mathrm{i}\alpha}|\alpha\rangle$. Show that the determination of $\alpha_n \equiv \langle n|\alpha\rangle$ reduces to the Anderson model with disorder in site energies. Is the disorder truly random – do you think it should matter? To do this, it will help to make the following substitutions:

$$|\beta\rangle \equiv \left(1 + \mathrm{e}^{-\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}L^2/2}\right)|\alpha\rangle,$$
$$\mathrm{e}^{\mathrm{i}K\cos\theta} = \frac{1-\mathrm{i}D}{1+\mathrm{i}D}.$$

We have thus shown that the quantum kicked rotator is much like a disordered 1D system, whose precise behavior is extremely hard to determine exactly. An alternative viewpoint for the problem is given by the classical kicked rotator is a simple example of dynamics which transitions to chaos depending on the strength of the kicks, K.

(e) Justify that, if we let $\theta_n \equiv \theta(n^+)$ and $L_n \equiv \dot{\theta}(n^+)$, that

$$\theta_{n+1} - \theta_n = L_n,$$
$$L_{n+1} - L_n = K\sin\theta_n.$$

(f) Numerically investigate the behavior of the sequence θ_n. Show numerically that there is a critical value K_c, such that for $K > K_\mathrm{c}$ the sequence of θ_n appears chaotic, and provide a basic estimate of K_c.