The Trap Model

The trap model is a phenomenological model for the aging dynamics seen in some realistic glasses. The basic idea is as follows: the system is described by a continuum of states with possible energies $E \geq 0$, which are distributed according to the following PDF:

$$\mathrm{p}(E) = \mathrm{e}^{-E}.$$

The system jumps out of a state with energy E at some rate $\mathrm{e}^{-E/T}$, where T is the temperature of the system, and will fall in a random state drawn from the energy distribution above. In this problem, our goal will be to understand the behavior of the correlation function

$$C(t+\tau, t) \equiv \mathrm{P}(E(t+\tau) = E(t)).$$

i.e., the probability that the particle has not jumped between between times t and $t+\tau$.[1]

Let's begin by assuming that $T > 1$. For simplicity, let us assume that the limit

$$C(\tau) \equiv \lim_{t\to\infty} C(t+\tau, t)$$

exists: with a bit more work, this can actually be shown.

(a) First, compute the PDF $\mathrm{p}_{\mathrm{eq}}(E)$, which determines the probability that, at time t, the energy of the particle is E. You should see something "strange" happen as $T \to 1$.

(b) Given this result, compute $C(\tau)$, and show that it decays algebraically:

$$C(\tau) \sim \frac{1}{\tau^{T-1}}.$$

Thus, even for very large T, this model still exhibits aging behavior. Give a simple explanation for this.

Now, let us assume that $T < 1$. Our work above suggests that there is no stationary distribution, and we will see the dramatic consequences of this at the end of this problem. However, first we need to develop a new technique.

(c) Let $\rho(E, t)$ be the PDF at time t for finding the system at energy E. Argue that ρ obeys the differential equation

$$\partial_t \rho(E,t) = \left[\int_0^\infty \mathrm{d}E'\ \rho(E',t)\mathrm{e}^{-E'/T}\right]\mathrm{e}^{-E} - \mathrm{e}^{-E'/T}\rho(E,t)$$

(d) Consider the ansatz

$$\rho = u\phi(u)$$

where

$$u \equiv \frac{\mathrm{e}^{E/T}}{t}.$$

[1] Almost surely the particle will not, in a finite number of jumps, return to the same energy state.

Plug this into the equation of part (c). Show that you get

$$u^2 \frac{\mathrm{d}\phi}{\mathrm{d}u} + (u-1)\phi = t\text{-dependent source term},$$

and that if the t-dependent source term is to actually be independent of t, then $\phi \sim u^{-T}$ for small u.

(e) We'll also need the behavior of ϕ at large u. Argue that this scaling can be found by solving

$$u^2 \frac{\mathrm{d}\phi}{\mathrm{d}u} + (u-1)\phi = -u^{-T}$$

and subsequently solve this equation to determine the scaling behavior of ϕ at large u.

(f) Now, by whatever means you would like, compute the asymptotic behavior of $C(t+\tau, t)$ in the limit where $\tau \gg t$, and show that

$$C(t+\tau, t) \sim \left(\frac{t}{\tau}\right)^T.$$

For all waiting times τ, this result depends on the start time t. What is the physical reason this happens?

This sort of separation with 2 distinct aging phases is in fact seen experimentally in many glasses. This was the original motivation for this model, in fact.