Electrostriction

If we write the thermodynamic relation for a dielectric medium as

$$\mathrm{d}\mathcal{F} = \mathrm{d}\mathcal{F}_{\mathrm{t}} + \mathbf{E}\cdot\mathrm{d}\mathbf{D},$$

we are including the total free energy of the field, even in the absence of any material present. Here, $\mathcal{F}_{\mathrm{t}}$ just represents the "thermodynamic" component of the free energy density (not related to electromagnetism).

For many purposes, this is a large inconvenience. It would often be nicer to write the thermodynamic potentials in terms of a variable $\mathbf{E}_0$, the electric field that would be present in the absence of any material. Assume that we have some finite region of space consisting of dielectric material, and no free charges. We are interested in the modified free energy

$$F' = \int \left(\mathcal{F} - \frac{1}{2}\epsilon_0 {\mathbf{E}_0}^2\right)\mathrm{d}V.$$

(a) Show that

$$\mathrm{d}F' - \mathrm{d}F_{\mathrm{t}} = \int \left((\mathbf{D} - \epsilon_0\mathbf{E}_0)\cdot\mathrm{d}\mathbf{E}_0 + \mathbf{E}\cdot(\mathrm{d}\mathbf{D} - \epsilon_0\mathrm{d}\mathbf{E}_0) - \mathbf{P}\cdot\mathrm{d}\mathbf{E}_0\right)\mathrm{d}V.$$

(b) Assume that $\mathbf{E}_0$ is approximately uniform in the dielectric. Show that

$$\mathrm{d}F' - \mathrm{d}F_{\mathrm{t}} = -\mathbf{p}\cdot\mathrm{d}\mathbf{E}_0$$

where $\mathbf{p}$ is the net dipole moment of the material. To do this, you should use the fact that in electrostatics, electric fields may be written as the gradient of a potential, to remove most of the terms from the integral in part (a).

This approximate treatment is very useful when dealing with the following types of problems: suppose we have a chunk of dielectric material of initial volume V_0, held at temperature T. How much heat do we transfer to this object by turning on an electric field? To answer this question, it is convenient to work with the Gibbs free energy:

$$\mathrm{d}G = -S\mathrm{d}T + V\mathrm{d}P - \mathbf{p}\cdot\mathrm{d}\mathbf{E}_0.$$

Assume that our material obeys the following reasonable properties:

$$\begin{aligned}
\mathbf{p} = \mathbf{P}V &\approx \epsilon_0\chi(T)\mathbf{E}_0 V,\\
\epsilon_0\chi(T) &= \frac{\gamma}{T},\\
\frac{\partial V(T,P,\mathbf{E}_0)}{\partial P} &= -\kappa V,\\
\frac{\partial V(T,P,\mathbf{E}_0)}{\partial T} &= \beta V.
\end{aligned}$$

The constants β, κ and γ are independent of any parameters of the problem.

(c) Show that the heat transferred to the dielectric by turning on an electric field of strength $\mathbf{E}_0$ at constant temperature and pressure is

$$Q = \frac{1}{2}{\mathbf{E}_0}^2 T\frac{\partial}{\partial T}\left(\epsilon_0\chi(T)V\right).$$

(d) Show that Q has qualitatively behavior depending on whether T is large or small (quantify what this means).

(e) Calculate the change in volume, ΔV, associated with turning the electric field on. The deformations of a body by external electric fields are referred to as **electrostriction.** Assume χ is independent of pressure P.