4D Chern-Simons Electromagnetism

In this problem, we will consider a Lorentz-violating theory of electromagnetism, with Lagrangian given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}q^{\mu}A^{\nu}F^{\rho\sigma}.$$

Treat q^{μ} as a constant vector in spacetime. Since q^{μ} picks out a preferred direction in spacetime, we break the manifest Lorentz invariance of our theory. The latter term is called a Chern-Simons term, and in 3D theories, the term $\epsilon_{\mu\nu\rho}A^{\mu}F^{\nu\rho}$ is an allowed term in a Lorentz-invariant Lagrangian which has applications in condensed matter theories.

(a) Show that the action is gauge invariant under $A_{\mu} \to A_{\mu} + \partial_{\mu}\lambda$ if all fields vanish at infinity.

(b) Find the equations of motion in terms of the physical fields $\mathbf{E}$ and $\mathbf{B}$, and comment on the results. What does the Chern-Simons term look like?

(c) Now, consider a propagating plane wave. Show that there are two modes of propagation with dispersion relation given by

$$k^{\alpha}k_{\alpha}k^{\beta}k_{\beta} = k^{\alpha}k^{\beta}q_{\alpha}q_{\beta} - k^{\alpha}k_{\alpha}p^{\beta}p_{\beta}.$$

(d) Suppose that the vector Q is spacelike: $Q^{\mu} = (0, 0, 0, Q)$. Sketch the dispersion relations in this case.

(e) Suppose that the vector Q is timelike: $Q^{\mu} = (Q, 0, 0, 0)$. Show that the dispersion relation is

$$\omega^2 = k(k \pm Q).$$

Discuss what happens at long wavelengths.

(f) Now, consider the case of the previous part. Suppose that two waves of the same wavelength in opposite modes travel the same distance L, assuming that they start in phase. What is the phase difference between these two modes after traveling a distance L, if ω and k are small?

Using the result of part (f) and some astrophysical data, incredibly tiny bounds can be put on the value of Q.